MATRIX-GEOMETRIC SOLUTION OF A SINGLE SERVER QUEUE WITH GROUP ARRIVALS AND PHASE TYPE SERVICE

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BSTRACT

A matrix geometric solution to the $M^{[x]}/PH/1$ is demonstrated. The essential step is the calculation of a matrix R, the unique non-negative solution of a quadratic matrix equation. Though the main structure of the Markov chain is of M/G/1 type, the assumption of bounded group arrivals allows handling the problem, after aggregation of states, as a structured Markov chain of GI/M/1 type and thus matrix geometric solution of the queue length distribution is admissible. Explicit equations are derived for the mean system length, the mean queue length, and the mean group waiting time. Finally, we discuss the virtual waiting time distribution of a group. The basic result of this paper is that structured Markov chains of M/G/1 type with finite bandwidth can be equivalently studied as GI/M/1 type chains after a suitable aggregation of states is done. The main disadvantage is the large dimension of matrices encountered in the computation.

NTRODUCTION

Customers arrive in groups to a single service facility ccording to a Poisson process with rate λ. Consecutive oup sizes are independent, bounded by N and have the mmon probability density $\{\theta_i\}$, $1 \leq i \leq N$. The service cility serves one customer at a time and the service time stribution is a PH-type (see this paper). This queue is noted as the $M^{[x]}/PH/1$.

he $M^{[x]}/PH/1$ queue has been treated many times in literature. Neuts [1] developed a recursive technique compute the steady state queue length probabilities nsidering both the discrete and continuous phase type tributions. Chaudhury and Templeton [2] considered the $^{x]}/E_K/1$ queue using a generating function approach. n Hoorn [3] considered the $M^{[x]}/G/1$ queue using the enerative approach and obtained steady state ressions for the mean queue length probability using a ursive technique in the case of phase type distributions. approach in this paper seems to be the most nising as the group distribution may well be state endent. Altiok [4] considered the $M^{[x]}/C_K/1$ queue g a recursive technique, however his work can be idered as a special case of the work done by Neuts Elsayed [5] used a generating function-algorithmic oach for the $M^{[x]}/C_K/1$ and obtained the steady state e length distribution for the important special cases $_2$, $E_{J,K}$ (mixture of Erlang distributions) and the deterministic distribution as special cases. Steady state probabilities are computed recursively exploiting the obtained generating function. Also mean system size for the cases considered are obtained. An algorithm for the general Coxian distribution is also proposed.

We should note that the queue $M^{[x]}/PH/1$ is a special case of the queue with versatile Markovian point process input and general service analyzed by Ramaswami [6].

In this work, we consider the $M^{[x]}/PH/1$ using matrix-geometric solution approach Neuts [7]. The assumption of Group arrivals which are independent, upper bounded by N and consecutive arrivals have the common probability density, is realistic in many cases such as the degree of parallelism in a computer system, the maximum number of packets that a message can be split into in a switching node,... etc. This assumption renders the solution of this queue as an M/PH/1-like queue. The special properties of this well behaved queue can be exploited to allow a semi-explicit solution for this queuing system. Although the structure of this queue is mainly of the M/G/1 type which has a different approach for analysis (see Neuts [8]), the beauty of the matrix geometric solution provides semi-explicit analytic expressions for many of the system performance measures in terms of a computable (in most cases) matrix R.

PHASE TYPE DISTRIBUTION REPRESENTATION

A phase type distribution F(.) with representation ($\underset{\sim}{\alpha}$,T) is the distribution of time till absorption in the (K+1) state Markov process with generator

$$Q = \begin{vmatrix} T & \underset{\sim}{T}^0 \\ \underset{\sim}{0} & 0 \end{vmatrix}$$

and initial probability vector ($\underset{\sim}{\alpha}$, α_{K+1}), without loss of generality we can assume that $\alpha_{K+1}=0$ throughout this work and that $\underset{\sim}{\alpha}$ is a true probability (row) vector. This distribution corresponds to K exponential stages with general feed forward and feed backward connections between the stages. Furthermore, the parameter of each of the exponential stages is not equal in general. The square matrix T is non-singular, has negative diagonal elements, non-negative off-diagonal elements and satisfies

$$T\underset{\sim}{e} + \underset{\sim}{T}^0 = \underset{\sim}{0},$$

where $\underset{\sim}{e}$ is a K-column vector with all components equal to one. Throughout this work $\underset{\sim}{e}$ is assumed to be a vector with all its components equal to one with an appropriate dimension to match the case under consideration.

We further assume that the matrix $T + \underset{\sim}{T}^0.\underset{\sim}{\alpha}$ is irreducible. The representation ($\underset{\sim}{\alpha}$, T) is then said to be irreducible.

The continuous probability distribution F(.) is itself given by

$$F(x) = 1 - \underset{\sim}{\alpha}\ \exp(Tx)\ \underset{\sim}{e}, \text{ for } x >= 0,$$

The n^{th} moment of the time till absorption in this CTMC is given by

$$\mu_n = -\underset{\sim}{\alpha}\ (T^{-1})^n\ \underset{\sim}{e}$$

The well known generalized Erlang and Coxian distribution of order K are special cases of continuous time phase distribution, the representation of these known types are respectively:

$$T = \begin{vmatrix} -\mu_1 & \mu_1 & & \\ & -\mu_2 & \mu_2 & \\ & \cdot & \cdot & \cdot \\ & & & -\mu_K \end{vmatrix}, \underset{\sim}{T}^0 = \begin{vmatrix} 0 \\ 0 \\ \vdots \\ \mu_K \end{vmatrix}$$

and $\underset{\sim}{\alpha}$ = (1 0 ... 0) with $\mu_j >= 0,\ 1 <= j <= K$, and

$$T = \begin{vmatrix} -\mu_1 & a_1\mu_1 & & \\ & -\mu_2 & a_2\mu_2 & \\ & \cdot & \cdot & \cdot \\ & & & -\mu_K \end{vmatrix}, \underset{\sim}{T}^0 = \begin{vmatrix} b_1\mu_1 \\ b_2\mu_2 \\ \vdots \\ \mu_K \end{vmatrix}$$

and $\underset{\sim}{\alpha}$ = (1 0 ... 0)

A complete discussion on phase type probability distributions can be found in Neuts [9] and Neuts [7].

THE $M^{[x]}$/PH/1 QUEUE

The underlying Markov chain of the $M^{[x]}$/PH/1 queue can be either a continuous or a discrete time Markov chain depending on the service distribution representation whether continuous or discrete phase type distribution or density respectively. In this work we consider only the continuous parameter phase distribution, which has, as special cases, Erlang, Generalized Erlang and Coxian distributions.

The state space of this markov process is

$$S = \{0,(n,j): n >= 1, 1 <= j <= K\},$$

the state 0 corresponds to the empty system and (n,j) is the state that there are n units in the queuing system and that the unit currently receiving service is in fictitious stage j.

To allow a matrix geometric solution for this queuing system, we consider group arrivals which are independent, upper bounded by N and consecutive arrivals have the common probability density θ_i, $1 <= i <= N$, the stationary probability that the arriving group is of size i. The mean group size is denoted by θ_1 is assumed to be finite and is easily calculated from

$$\theta_1 = \sum_{i=1}^{N} i\,\theta_i.$$

The assumption of bounded group size can be made realistic for distributions that are not upper bounded by calculating the integer N-1 for which $\theta_{N-1} < \epsilon$, where ϵ is small (typically in the range 1e-3 to 1e-6) and then setting

$$\theta_N = 1 - \sum_{i=1}^{N-1} \theta_i.$$

Let P_o be the stationary probability of the empty system, and $P(n,j)$ be the stationary probability that our Markov process is in state (n,j). For $n > 0$, define $\underline{P}_n$, a row vector of dimension K, to be $(P(n,1)\ P(n,2)\ \ldots\ P(n,K))$.

We can write the stationary (steady state) Chapman Kolmogrov equations for the $M^{[x]}/PH/1$ with phase representation $(\underline{\alpha}, T)$ as follows:

$$\lambda P_o = \underline{P}_1 \underline{T}^0 \tag{1}$$

$$\lambda\theta_1 P_o \underline{\alpha} + \underline{P}_1(T-\lambda I) + \underline{P}_2 \underline{T}^0.\underline{\alpha} = 0, \tag{2}$$

For $1 <= n <= N$,

$$\lambda\theta_n P_o \underline{\alpha} + \lambda\sum_{i=1}^{n-1} \theta_{n-i} \underline{P}_i + \underline{P}_n(T-\lambda I) + \underline{P}_{n+1} \underline{T}^0.\underline{\alpha} = 0 \tag{3}$$

and finally for $n > N$

$$\lambda\sum_{i=n-N}^{n-1} \theta_{n-i} \underline{P}_i + \underline{P}_n(T-\lambda I) + \underline{P}_{n+1} \underline{T}^0.\underline{\alpha} = 0 \tag{4}$$

Let

$$\gamma_i = \lambda\ \theta_i,\ 1 <= i <= N \tag{5}$$

Then the infinitesimal generator Q for this Markov process can be written as follows:

Define the following quantities:

$$M = N \times K, \tag{7}$$

$$\underline{a}_o = (\gamma_1\underline{\alpha}\ \ \gamma_2\underline{\alpha}\ \ \ldots\ \ \gamma_N\underline{\alpha}) \tag{8}$$

which is an M-row vector,

$$\underline{a}_2 = \begin{vmatrix} \underline{T}^0 \\ \underline{0} \\ \vdots \\ \underline{0} \end{vmatrix} \tag{9}$$

which is an M-column vector

$$A_o = \begin{vmatrix} \gamma_N I & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \gamma_{N-1} I & \gamma_N I & 0 & 0 & \cdots & \cdots & 0 \\ \gamma_{N-2} I & \gamma_{N-1} I & \gamma_N I & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & & & \vdots \\ \gamma_1 I & \gamma_2 I & \gamma_3 I & \cdots & \cdots & \cdots & \gamma_N I \end{vmatrix} \tag{10}$$

$$\underline{Q} = \begin{array}{c|ccccccccc|} & 0 & 1 & 2 & \cdots & N-1 & N & N+1 & N+2 & \cdots \\ 0 & -\lambda & \gamma_1\underline{\alpha} & \gamma_2\underline{\alpha} & \cdots & \gamma_{N-1}\underline{\alpha} & \gamma_N\underline{\alpha} & \underline{0} & \underline{0} & \underline{0}\ \cdots \\ 1 & \underline{T}^0 & T-\lambda I & \gamma_1 I & \cdots & \gamma_{N-2} I & \gamma_{N-1} I & \gamma_N I & 0 & 0\ \cdots \\ 2 & \underline{0} & \underline{T}^0.\underline{\alpha} & T-\lambda I & \cdots & \gamma_{N-3} I & \gamma_{N-2} I & \gamma_{N-1} I & \gamma_n I & 0\ \cdots \\ 3 & \underline{0} & 0 & \underline{T}^0\underline{\alpha} & \cdots & \gamma_{N-4} I & \gamma_{N-3} I & \gamma_{N-2} I & \gamma_{N-1} I & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \\ N-1 & & & & & T-\lambda I & \gamma_1 I & \gamma_2 I & \gamma_3 I & \cdots \\ N & & & & & \underline{T}^0\underline{\alpha} & T-\lambda I & \gamma_1 I & \gamma_2 I & \cdots \\ N+1 & & & & & 0 & \underline{T}^0\underline{\alpha} & T-\lambda I & \gamma_1 I & \cdots \\ \vdots & & & & & \vdots & \vdots & \vdots & \vdots & \\ \vdots & & & & & \vdots & \vdots & \vdots & \vdots & \end{array} \tag{6}$$

which is a lower diagonal matrix either on the block or element level.

$$A = \begin{vmatrix} T-\lambda I & \gamma_1 I & \gamma_2 I & \cdots & & \cdots & \gamma_{N-1} I \\ T^0.\underset{\sim}{\alpha} & T-\lambda I & \gamma_1 I & \cdots & & \cdots & \gamma_{N-2} I \\ 0 & T^0.\underset{\sim}{\alpha} & T-\lambda I & \cdots & & \cdots & \cdot \\ 0 & 0 & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \cdot \\ & & & & T^0.\underset{\sim}{\alpha} & T-\lambda I & \gamma_1 I \\ 0 & 0 & 0 & & 0 & T^0\underset{\sim}{\alpha} & T-\lambda I \end{vmatrix} \quad (11)$$

which is an upper Hessenberg matrix of dimension M, and

$$A_2 = \begin{vmatrix} 0 & 0 & \cdots & T^0.\underset{\sim}{\alpha} \\ 0 & 0 & \cdots & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & 0 & \cdots & 0 \end{vmatrix} \quad (12)$$

which is a singular square matrix of dimension M, and is highly sparse as only one block of size N x N is generally nonzero.

After defining the above quantities, and by aggregation of each of N states (i,j) starting from i=1, into a single state the matrix Q can be rewritten in a highly structured form as follows:

$$Q = \begin{vmatrix} -\lambda & \underset{\sim}{a}_0 & 0 & 0 & 0 & \cdots \\ \underset{\sim}{a}_2 & A_1 & A_o & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_o & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_o & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix} \quad (13)$$

Let the corresponding probability vector $\underset{\sim}{X}$ of that continuous time Markov chain be partitioned into the vectors x_o, $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, ... where

$$x_o = P_o \quad (14)$$

$$\underset{\sim}{x}_1 = (\underset{\sim}{P}_1 \ \underset{\sim}{P}_2 \ \ldots\ldots \ \underset{\sim}{P}_N)$$

$$\underset{\sim}{x}_i = (\underset{\sim}{P}_{(i-1)xN+1} \ \underset{\sim}{P}_{(i-1)xN+2} \quad \underset{\sim}{P}_{iN}) \quad (15)$$

Clearly, $\underset{\sim}{x}_i$, i >= 1 is an M-row vector. The Markov chain resulting from this aggregation of states can be described for each level i by the pair (n,j), where n $\in$ { (i-1)*N+1, (i-1)*N+2, ..., i*N }, 1 <= j <= K. The set (n,j) so constructed will be called level i.

The structured Markov chain given by equation (13) can be solved using the standard matrix-geometric method developed mainly by Neuts as well as G. Latouche, D. Lucantoni, V. Ramaswami and other contributors. The main results, theorems, algorithms, numerical experience, and many applications can be found in Neuts [7]. Here, we briefly mention the main results for sake of completeness, and the interested reader should consult the reference cited at the end of this paper and the huge bibliography in Neuts [8].

The following two theorems are proved in Neuts [7].

Theorem 1: Consider a positive recurrent, continuous time Markov chain with infinitesimal generator Q of the form

$$Q = \begin{vmatrix} B_o & A_o & 0 & 0 & 0 & \cdots \\ B_1 & A_1 & A_o & 0 & 0 & \cdots \\ B_2 & A_2 & A_1 & A_o & 0 & \cdots \\ B_3 & A_3 & A_2 & A_1 & A_o & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix} \quad (16)$$

the elements of Q are M x M matrices. The off diagonal elements of Q are non-negative, the diagonal elements are negative and

$$\sum_{v=0}^{K} A_v \underset{\sim}{e} + B_K \underset{\sim}{e} = 0, \quad K \geq 0,$$

then the steady state probability distribution ($\underset{\sim}{x}_o$, $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, ...) of this system is matrix geometric in the sense that $\underset{\sim}{x}_n = \underset{\sim}{x}_{n-1} R$, where R is an M x M matrix whose spectral radius is less than one and which is the minimum non-negative solution of the nonlinear matrix equation

$$\sum_{K=0}^{\infty} R^K A_K = 0$$

Theorem 2: Define the matrix $B[R] = \sum_{K=0}^{\infty} R^K B_K$, then the irreducible Markov process with infinitesimal generator given by (16) is positive recurrent if and only if R has all its eigenvalues inside the unit disk and if there exists a positive vector $\underset{\sim}{x}_o$ such that $\underset{\sim}{x}_o B[R] = \underset{\sim}{0}$, moreover the stationary probability vector $\underset{\sim}{X}$ satisfying $\underset{\sim}{X} Q = \underset{\sim}{0}$, and $\underset{\sim}{X} \underset{\sim}{e} = 1$ is given by $\underset{\sim}{X} = (\underset{\sim}{x}_o, \underset{\sim}{x}_o R, \underset{\sim}{x}_o R, ...)$ where $\underset{\sim}{x}_o$ is the solution of

$$\underset{\sim}{x}_o B[R] = \underset{\sim}{0}, \text{ and } \underset{\sim}{x}_o [I-R]^{-1} \underset{\sim}{e} = 1$$

Applying the above theorems to our $M^{[x]}/PH/1$ queue, we notice that the infinitesimal generator of our queue is a simpler version of the one defined above, as only the matrices A_o, A_1, and A_2 exist and all other A's are equal to zero. This generator is of the Quasi Birth-Death Process type which exhibit more structured properties than the general case. There is another difference in our queue in that the probability $\underset{\sim}{x}_o$ is not a vector of dimension M, but is a scalar quantity. We form the steady state probability distribution for this system as follows:

The Equilibrium equations for this QBD process are

$$-\lambda x_o + \underset{\sim}{x}_1 \underset{\sim}{a}_2 = 0, \text{ i.e., } x_o = \underset{\sim}{x}_1 \underset{\sim}{a}_2 / \lambda \tag{17}$$

$$x_o \underset{\sim}{a}_o + \underset{\sim}{x}_1 A_1 + \underset{\sim}{x}_2 A_2 = \underset{\sim}{0} \tag{18}$$

$$\underset{\sim}{x}_{i-1} A_o + \underset{\sim}{x}_i A_1 + \underset{\sim}{x}_{i+1} A_2 = \underset{\sim}{0} \tag{19}$$

Then the steady state solution of this QBD process is given by

$$\underset{\sim}{x}_n = \underset{\sim}{x}_{n-1} R, \; n > 1, \tag{20}$$

where the M×M matrix R is the minimal non-negative solution of the quadratic matrix equation

$$A_o + R A_1 + R^2 A_2 = 0 \tag{21}$$

this equation can be written in the following form

$$R = A_o [-A_1]^{-1} + R^2 A_2 [-A_1]^{-1}$$

The above equation can be used to evaluate R iteratively, as follows. Let

$$R_{(o)} = A_o [-A_1]^{-1}, \text{ and}$$

$$R_{(n+1)} = A_o [-A_1]^{-1} + R^2_{(n)} A_2 [-A_1]^{-1}, \; n \geq 0$$

The above iterative scheme is in general rapidly convergent because of the special properties of the matrix R.

Because of the special structure of the matrix A_2 we can exploit the result developed by Gillenet and Latouche to find the matrix R explicitly.

The matrix A_2 can be written as $\underset{\sim}{a}_2 . \underset{\sim}{w}$ where $\underset{\sim}{a}_2$ is defined in (8) and

$$\underset{\sim}{w} = (0\; 0 \; ... \; \underset{\sim}{\alpha}), \tag{22}$$

clearly $\underset{\sim}{w}\, \underset{\sim}{e} = 1$. Then the following theorem can be directly applied to find the matrix R explicitly and to relate the vector $\underset{\sim}{x}_1$, to $\underset{\sim}{x}_o$. In all single server queuing systems with infinite waiting room the probability of the empty system is 1-ρ, where ρ is the queue utilization.

Theorem 3: (Theorem 2 page 152 F. Gillent and G. Latouche [10])

If the matrix $A_2 = \underset{\sim}{a}_2 . \underset{\sim}{w}$, and $\underset{\sim}{w}\, \underset{\sim}{e} = 1$, and if the Markov chain given by (13) is ergodic, then the matrix R is given by

$$R = -A_o Z^{-1}, \tag{23}$$

where

$$Z = A_1 + A_o \underset{\sim}{e} . \underset{\sim}{w} \tag{24}$$

Moreover the vector $\underset{\sim}{x}$ satisfies the relation

$$\underset{\sim}{x}_i = -\underset{\sim}{x}_o \underset{\sim}{a}_2 Z^{-1} R^{i-1} \text{ for } i \geq 1, \tag{25}$$

where

$$x_o = 1 - \rho \tag{26}$$

where ρ the queue utilization, is equal $\lambda\theta\mu_1$, where θ is the mean group size and μ_1 is the mean service time of the service distribution given by $-\underset{\sim}{\alpha}\, T^{-1} \underset{\sim}{e}$.

In the following sections we obtain analytic closed form expressions for the mean system size and the mean queue size, in terms of the $\underset{\sim}{x}_1$ vector and the matrix R.

PERFORMANCE MEASURES OF THE $M^{[x]}/PH/1$

Mean System Size of the $M^{[x]}/PH/1$

Theorem 4: The mean system size L is given by

$$\underset{\sim}{x}_1 (I - R)^{-1} [\, N (I - R)^{-1} R \underset{\sim}{e} + \underset{\sim}{U} \,] \tag{27}$$

The second moment of the mean system size $L^{(2)}$ is given by

$$\underset{\sim}{x}_1 (I-R)^{-1} [N(I-R)^{-1} (N[2(I-R)^{-1} R + I] R \underset{\sim}{e} + 2R\underset{\sim}{U}) + \underset{\sim}{U} \bullet \underset{\sim}{U}] \tag{28}$$

where

$$\underline{U} = (\underline{v}_1 \ \underline{v}_2 \ \dots \ \underline{v}_N)^T \tag{29}$$

which is an M-column vector and

$$\underline{v}_i = (i \ i \ \dots \ i) \tag{30}$$

is a k-row vector.

proof:

The steady state solution vector $\underline{X}$ is related to the steady state system size probability $\underline{P}$ by the following relation.

$$x_o = P_o, \ \underline{x}_i = (\underline{P}_{(i-1)K+1} \ \underline{P}_{(i-1)K+2} \ \cdots \ \underline{P}_{iK})$$

The mean system size is

$$L = \Big(\sum_{n=1}^{\infty} n\underline{P}_n\Big)\underline{e}_1,$$

where $\underline{e}$ is a K-column vector with all entries equal to one. The vectors $\underline{P}_n$ can not be expressed explicitly in terms of the matrix R, and $\underline{x}_1$. Define the vector $\underline{U}$ as given by (29) which is an M-column vector. Then L can be written in the form

$$L = \sum_{i=1}^{\infty}\underline{x}_1 R^{i-1}((i-1)N\underline{e} + \underline{U})$$

$$= \underline{x}_1\Big[N\Big(\sum_{i=1}^{\infty} iR^i\Big)\underline{e} + \Big(\sum_{i=1}^{\infty} R^i\Big)\underline{U}\Big]$$

$$= \underline{x}_1\big[N(I-R)^{-2}R\underline{e} + (I-R)^{-1}\underline{U}\big]$$

$$= \underline{x}_1(I-R)^{-1}\big[N(I-R)^{-1}R\underline{e} + \underline{U}\big]$$

We can also compute the second moment of the queue length as follows:

$$L^{(2)} = \Big(\sum_{n=1}^{\infty} n^2\underline{P}_n\Big)\underline{e}$$

$$L^{(2)} = \sum_{i=1}^{\infty}\underline{x}_1 R^{i-1}(i-1)N\underline{e} + \underline{U}) \bullet ((i-1)N\underline{e} + \underline{U})$$

where $\bullet$ is the shur or entry wise product of vectors.

$$L^{(2)} = \underline{x}_1\Big[N^2\Big(\sum_{i=0}^{\infty} i^2R^i\Big)\underline{e} + 2N\Big(\sum_{i=0}^{\infty} iR^i\Big)\underline{U} + \Big(\sum_{i=0}^{\infty} R^i\Big)\underline{U}\bullet\underline{U}\Big]$$

$$= \underline{x}_1\big[N^2[2(I-R)^{-3}R^2 + (I-R)^{-2}R]\underline{e} + 2N(I-R)^{-2}R\underline{U}(I-R)^{-1}\underline{U}\bullet\underline{U}\big]$$

$$= \underline{x}_1(I-R)^{-1}\big[N(I-R)^{-1}(N[2(I-R)^{-1}R + I]R\underline{e} + 2R\underline{U}) + \underline{U}\bullet\underline{U}\big]$$

The mean queue length

Theorem 5: The mean queue length L is given by

$$\underline{x}_1(I-R)^{-1}\,[N(I-R)^{-1}R\underline{e} + \underline{Y}\,] \tag{31}$$

where

$$\underline{Y} = (\underline{v}_o \ \underline{v}_1 \ \cdots \ \underline{v}_{N-1}) \tag{32}$$

where $\underline{v}_i$, $0 <= i <= N$ is defined by (30), and the second moment of the queue length $L_q^{(2)}$ is given by

$$\underline{x}_1(I-R)^{-1}\big[N(I-R)^{-1}(N[2(I-R)^{-1}R + I]R\underline{e} + 2R\underline{Y}) + \underline{Y}\bullet\underline{Y}\big] \tag{33}$$

proof:

The mean queue length is given by

$$L_q = \Big(\sum_{n=1}^{\infty}(n-1)\underline{P}_n\Big)\underline{e}_1 \text{ and } L_q^{(2)} = \Big(\sum_{n=1}^{\infty}(n-1)^2\underline{P}_n\Big)\underline{e}_1$$

and then proceeding as done theorem 4 after using the vector $\underline{Y}$ instead of $\underline{U}$ in each step of the proof, we obtain the above results.

The mean group waiting time

The virtual waiting time of the group is the time spent in the queue by a group arriving at an arbitrary point till the first customer in the group is admitted to the service facility. The time the first customer has to wait may be viewed as the time till absorption in the Markov chain with generator

$$\begin{array}{c|cccccccc|}
0 & 0 & 0 & 0 & 0 & . & . & .... & \\
1 & \underline{a}_2 & D & 0 & 0 & . & . & .... & \\
2 & 0 & A_2 & D & 0 & 0 & . & .... & \\
3 & 0 & 0 & A_2 & D & 0 & 0 & .... & \\
 & . & . & . & . & . & . & .... & \\
 & . & . & . & . & . & . & .... & \\
 & . & . & . & . & . & . & .... &
\end{array}$$

where the matrix D is obtained by replacing λ in A_1 by zero. if a group arrives it will find the system in state (n,j), where

$(i-1)*N + 1 <= n <= i*N$ and $1 <= j <= K$, with probability $\underline{x}_i = \underline{x}_1 R^{i-1}$, the time till absorption in this Markov chain will have an LST given by $[(sI-D)^{i-1}A_2]^{i-1}(sI-$

$D)^{-1}\underset{\sim}{a}$, hence the virtual waiting time of a group $W_1(t)$ will have an LST given by

$$w_1(s) = x_o + \sum_{n=1}^{\infty} \underset{\sim}{x}_n \left[(sI-D)^{-1}A_2\right]^{n-1}(sI-D)^{-1}\underset{\sim}{a}_2$$

post-multiplying the last equation by $\underset{\sim}{w}\,\underset{\sim}{e} = 1$, and observing that $A_2 = \underset{\sim}{a}_2.\underset{\sim}{w}$, then

$$w_1(s) = x_o + \sum_{n=1}^{\infty} \underset{\sim}{x}_n \left[(sI-D)^{-1}A_2\right]^{n}\underset{\sim}{e} \tag{34}$$

and finally

$$w_1(s) = x_o + \sum_{n=1}^{\infty} \underset{\sim}{x}_n (sI-D)^{-1}\underset{\sim}{a}_2\left[\underset{\sim}{w}(sI-D)^{-1}\underset{\sim}{a}_2\right]^{n-1} \tag{35}$$

Upon differentiating the above equation the expected value for the mean virtual waiting time of a group can be found

$$W_1 = \frac{dw}{ds}\Big|_{s=0}$$

$$= -\mathcal{L}im_{s\to 0} \sum_{n=1}^{\infty} \underset{\sim}{x}_n[(sI-D)^{-1}(n-1)(\underset{\sim}{w}(sI-D)^{-1}\underset{\sim}{a}_2)^{n-2}.$$

$$(-\underset{\sim}{w}(sI-D)^{-2}\underset{\sim}{a}_2) - (sI-D)^{-2}(\underset{\sim}{w}(sI-D)^{-1}\underset{\sim}{a}_2)^{n-1}]\underset{\sim}{a}_2$$

Noting that

$$\mathcal{L}im_{s\to 0}(\underset{\sim}{w}(sI-D)^{-1}\underset{\sim}{a}_2) = (\underset{\sim}{w}(sI-D)^{-1}\underset{\sim}{a}_2) = (\underset{\sim}{w}\,\underset{\sim}{e}) = 1$$

and that

$$\mathcal{L}im_{s\to 0}(-\underset{\sim}{w}(sI-D)^{-2}\underset{\sim}{a}_2) = (-\underset{\sim}{w}(-D)^{-1}\underset{\sim}{a}_2) = (-\underset{\sim}{w}(-D)^{-1}\underset{\sim}{e})$$

then

$$W_1 = \sum_{n=1}^{\infty} \underset{\sim}{x}_n\left[(n-1)(\underset{\sim}{w}(-D)^{-1}\underset{\sim}{e})\underset{\sim}{e} + (-D)^{-1}\underset{\sim}{e}\right] \tag{36}$$

The quantities $(-D)^{-1}\underset{\sim}{e}$ and $\underset{\sim}{w}(-D)^{-1}\underset{\sim}{e}$ are now expressed in terms of the queue parameters in the following theorem

Theorem 6:

$$(-D)^{-1}\underset{\sim}{e} = \mu_1 \underset{\sim}{Y} + \underset{\sim}{h} \tag{37}$$

where μ_1 is the mean service rate of the server, $\underset{\sim}{Y}$ is defined by (32) and

$$\underset{\sim}{h} = \begin{vmatrix} -T^{-1}\underset{\sim}{e}_1 \\ -T^{-1}\underset{\sim}{e}_1 \\ \vdots \\ -T^{-1}\underset{\sim}{e}_1 \end{vmatrix}$$

and

$$w(-D)^{-1}\underset{\sim}{e} = N\mu_1 \tag{38}$$

Proof:

$$D = \begin{vmatrix} T & 0 & 0 \dots & \dots & 0 \\ T^0.\underset{\sim}{\alpha} & T & 0 \dots & \dots & 0 \\ 0 & T^0.\underset{\sim}{\alpha} & T \dots & \dots & 0 \\ . & 0 & & & \\ . & . & & & \\ . & . & & T & 0 \\ 0 & 0 & & T^0.\underset{\sim}{\alpha} & T \end{vmatrix}$$

Making elementary row operations on this matrix, we can find the inverse in the following form

$$D = \begin{vmatrix} T^{-1} & 0 & 0 & \dots & \dots & 0 \\ -T^{-1}\phi & T^{-1} & 0 & \dots & \dots & 0 \\ T^{-1}\phi^2 & -T^{-1}\phi & T^{-1} & \dots & \dots & 0 \\ . & . & & & & \\ . & . & & & & \\ . & . & & & T^{-1} & 0 \\ T^{-1}(-\phi)^{N-1} & T^{-1}(-\phi)^{N-2} & & & -T^{-1}\phi & T^{-1} \end{vmatrix} \tag{39}$$

where

$$\phi = \underset{\sim}{T}^0.\underset{\sim}{\alpha}\,T^{-1} \tag{40}$$

then $D^{-1}\underset{\sim}{e}$ takes the form

$$D^{-1}\underset{\sim}{e} = \begin{vmatrix} T^{-1}\underset{\sim}{e}_1 \\ -T^{-1}\phi\,\underset{\sim}{e}_1 + T^{-1}\underset{\sim}{e}_1 \\ . \\ . \\ T^{-1}(-\phi)^{N-1}\underset{\sim}{e}_1 + T^{-1}(-\phi)^{N-2}\underset{\sim}{e}_1 + \dots - T^{-1}\phi\underset{\sim}{e}_1 + T^{-1}\underset{\sim}{e}_1 \end{vmatrix} \tag{41}$$

but

This expression may be further simplified. We post-multiply equation (47) by $\underset{\sim}{e}$ and note that $D\underset{\sim}{e} + A_2\underset{\sim}{e} = 0$, then we obtain $V^0 A_2 \underset{\sim}{e} = (I-R)^{-1}\underset{\sim}{e}$, upon substituting in (50), we obtain

$$W_1(x) = 1 - \underset{\sim}{x}_1 \phi(x) A_2 \underset{\sim}{e}, \text{ for } x >= 0 \qquad (51)$$

The waiting time distribution can thus be obtained by evaluating the matrix V^0 and then solving the matrix-differential equation (48).

By a similar argument as in Neuts [12] we can obtain results for the asymptotic exponentiality (see Neuts [14]) of the waiting time distribution.

NUMERICAL EXAMPLES

The algorithmic approach presented in this paper was successfully programmed using the 'C' language and applied to many $M^{[x]}/PH/1$ queues and the results completely agree with cases that can be analyzed using generating functions (See El-Sayed [5]).

To demonstrate the usefulness of these algorithm we present below results for three queuing systems under various inputs.

The first queue is a Coxian of order two with the following representation:

$$T = \begin{vmatrix} -4 & 2.8 \\ & -3 \end{vmatrix}, \quad \underset{\sim}{T}^0 = \begin{vmatrix} 1.2 \\ 3 \end{vmatrix}, \quad \underset{\sim}{\alpha} = (1 \;\; 0)$$

The second queue is a mixture of Erlang of order 3 and Erlang of order 5 with the representation

$$T = \begin{vmatrix} -5 & 5 & & & \\ & -5 & 5 & & \\ & & -5 & 3 & \\ & & & -5 & 5 \\ & & & & -5 \end{vmatrix}, \underset{\sim}{T}^0 = \begin{vmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 5 \end{vmatrix}, \underset{\sim}{\alpha} = (1\;0\;0\;0\;0)$$

The third queue is a general phase type with the following representation

$$T = \begin{vmatrix} -5 & 0 & 1.5 & 2 \\ 1.8 & -3 & 0 & 1.2 \\ 0 & 2.7 & -4 & 0 \\ 1.1 & 0 & 1.7 & -5 \end{vmatrix}, \underset{\sim}{T}^0 = \begin{vmatrix} 1.5 \\ 0 \\ 1.3 \\ 2.2 \end{vmatrix}, \underset{\sim}{\alpha} = (.4\;.25\;.2\;.15)$$

The utilization is kept constant at 0.9 and the group size density is varied as shown in Table 1.

Note that the truncated geometric density with parameter P and upper bound N is constructed as follows:

$$P(i) = p\, q^{i-1},\; i = 1,2,\ldots,N-1$$

$$P(N) = q^{N-1}, \text{ where } q = 1 - p$$

The quantities obtained in each case are the mean queue length L_q and variance $Var(L_q)$, the mean waiting time of the first customer in a group W_1, the mean waiting time of an arbitrary customer within the arriving group W, the propagation constant of the tail of the probability distribution of the system population τ defined as

$$\lim_{n \to \infty} \frac{P_{n+1}}{P_n}$$

(see Neuts [14]), and the probability distribution of the system population.

CONCLUSION

A number of closed form expressions has been obtained for the $M^{[x]}/PH/1$ queue.

The main conclusion which deserves further study is that structured Markov chains of M/G/1 type which has finite bandwidth can be studied as chains of GI/M/1 type. The latter has a matrix-geometric solution. The main step is aggregation of states and construction of larger matrices which render the process as a QBD process. The main disadvantage is the large dimension of matrices encountered. By careful programming and specially written algorithms for handling sparse matrices, storage requirements can be reduced and speed of calculations can be achieved.

Further study of the equivalence of these two classes of Markov chains should be studied and verified.

Table 1. Group size densities.

Input	N	Density of θ	θ_1	θ_2
A	6	(0 0 0 0 0 1) constant batch size = 6	6	36
B	10	(0.1 0.1 ... 0.1)	5.5	38.5
C	10	truncated geometric with P=0.124	5.91857	46.62322
D	20	(0 0 0.3 0.4 0.2 0 0 ... 0 0.1)	5.5	54.1

Table 2.1. System performance for queue 1.

Input	Queue	L_q	$Var(L_q)$	W_1	W	τ
A	1	29.386	1031.5497	14.573	15.782	0.969449
B	1	33.885	1382.5393	16.748	18.196	0.973559
C	1	37.835	1717.3431	18.567	20.319	0.97624
D	1	46.650	2740.1424	22.917	25.053	0.981179

Table 2.2. Probability Distribution for queue 1.

Input	Queue	P(n<10)	P(n<20)	P(n<30)	P(n<40)	P(n<50)
A	1	0.3202	0.5019	0.6348	0.7322	0.8036
B	1	0.2963	0.4612	0.5878	0.6847	0.7588
C	1	0.2772	0.4304	0.5521	0.6478	0.7231
D	1	0.2648	0.3863	0.49358	0.5809	0.6535

Table 3.1. System performance for queue 2.

Input	Queue	L_q	$Var(L_q)$	W_1	W	τ
A	2	27.735	914.19423	23.786	25.886	0.967577
B	2	32.235	1247.0151	27.566	30.086	0.972170
C	2	36.183	1565.8766	30.882	33.771	0.975122
D	2	44.998	2553.0741	38.287	41.998	0.980507

Table 3.2. Probability Distribution for queue 2.

Input	Queue	P(n<10)	P(n<20)	P(n<30)	P(n<40)	P(n<50)
A	2	0.3306	0.5189	0.6541	0.7512	0.8211
B	2	0.3044	0.4749	0.6040	0.7014	0.7748
C	2	0.2837	0.4418	0.5661	0.6627	0.7378
D	2	0.2703	0.3947	0.5039	0.5923	0.6652

Table 4.1. System performance for queue 3.

Input	Queue	L_q	$Var(L_q)$	W_1	W	τ
A	3	30.7958	1137.5090	27.9461	30.149	0.970895
B	3	35.296	1504.0065	31.911	34.554	0.974642
C	3	39.244	1852.4140	35.390	38.420	0.977116
D	3	48.060	2905.5748	43.1569	47.050	0.981719

Table 4.2. Probability Distribution for queue 3.

Input	Queue	P(n<10)	P(n<20)	P(n<30)	P(n<40)	P(n<50)
A	3	0.3122	0.4884	0.6193	0.7167	0.7891
B	3	0.2901	0.4503	0.5748	0.6711	0.7456
C	3	0.2721	0.4213	0.5408	0.6357	0.7110
D	3	0.2604	0.3795	0.4852	0.57156	0.6438

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