DETERMINATION OF STIFFNESS AND DAMPING COEFFICIENTS IN JOURNAL BEARING WITH NON-NEWTONIAN LUBRICANT

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ABSTRACT

The stiffness and damping coefficients in journal bearing with non-Newtonian fluids as lubricants are calculated using the finite disturbance technique. The computation is based on a finite-difference approximation of a modified Reynolds equation for non-Newtonian fluid with power-law model. The resulting dimensionless linearized dynamic coefficients of the bearing with non-Newtonian fluids are presented in comparison with those of Newtonian lubricant.

NOMENCLATURE

C	radial clearance, m
$c_{\eta\eta}$, $c_{\eta\zeta}$, ...etc	damping coefficients N.s/m
$C_{\eta\eta}$, $C_{\eta\zeta}$, ... etc	dimensionless damping coefficients, $c_{\eta\zeta}C^3/\mu_r LR^3$
C_{xx}, C_{xy}, ... etc	dimensionless damping coefficients
D	journal diameter, m
e	eccentricity, m
$e_{i,j}$	rate of deformation
f_η, f_ζ	fluid forces in η, ζ directions respectively, N
F_η, F_ζ	dimensionless fluid forces in η, ζ directions respectively, $f_\eta C^2/\omega\mu_r LR^3$
f_e	unidirectional constant force, N
h	oil film thickness, m
H	dimensionless film thickness, h/C
$k_{\eta\eta}$, $k_{\eta\zeta}$, ... etc	stiffness coefficients, N/m
$K_{\eta\eta}$, $K_{\eta\zeta}$, ... etc	dimensionless stiffness coefficients, $k_{\eta\eta}C^3/\omega\mu_r LR^3$
L	bearing length, m
m	consistency constant
$2m_r$	rotor mass, kg
n	flow index
p	pressure, N/m^2
P	dimensionless pressure, $pC^2/\omega\ \mu_r\ R^2$
R	journal radius, m
t	time, s
u	velocity component along line of centers, m/s
v	velocity component normal to line of centers, m/s
U	dimensionless velocity, $u/\omega C$
V	dimensionless velocity, $v/\omega C$
w	total fluid film force, N
W	dimensionless total fluid film load, $wC^2/\omega\mu_r LR^3$
ϵ	eccentricity ratio e/C
θ	angular coordinate
μ_r	reference viscosity, PaS
$\hat{\eta}$, $\hat{\zeta}$, $\hat{\xi}$	coordinates
η, ζ, ξ	$\hat{\eta}/C$, $\hat{\zeta}/C$, $\hat{\xi}/L/2$ respectively, dimensionless coordinates
$\dot{\eta}$, $\dot{\zeta}$	$d\eta/d\tau$, $d\zeta/d\tau$ respectively, dimensionless
τ	$= \omega t$
$\tau_{i,j}$	shear stress tensor
ϕ	attitude angle
ω	angular velocity, rad/s

INTRODUCTION

It is well established that the action of oil film in journal bearings plays a major role in determining the dynamic characteristics of rotating machines. A defective oil film can cause the loss of rotor motion stability and the rise of self-excited vibrations.

The concept of stiffness and damping coefficients for journal bearings has proven a very powerful tool in modern rotor dynamic calculations for unbalance response, damped natural frequencies, and stability.

Stodola [1], Newkirch [2] and Hummel [3] originated the idea of representing the dynamic effect of the oil film by means of stiffness and damping coefficients. Their aim was to improve the calculation of the critical speeds of rotors.

However, this idea was not immediately accepted because of the non-linearity of load and displacement in journal bearings. But long experience and experiments have demonstrated the practical usefulness of the bearing dynamic coefficients when dealing with the dynamics of rotor-bearing systems. Some of the numerous theoretical and experimental works concerning the determination of stiffness and damping coefficients in different types of bearing using various approaches are given in the references [4,5,6,7,8,9,10,11].

Recently, hydrodynamically lubricated bearings have been more widely used under ever more severe conditions of speed, load and temperature. This lead to modern lubricating oils containing a large quantity of polymers as viscosity improver, hence their non-Newtonian behaviour. Various theories have been postulated to describe the flow behaviours of the non-Newtonian fluids. One is the "power law model", which gives the relation between the shear stress tensor τ_{ij} and the rate of deformation $e_{i,j}$ [12], [13] as:

$$\tau_{i,j} = m(|e_{i,j}|^2)^{\frac{n-1}{2}} e_{ij}$$

where m is the consistency constant and n is the flow behaviour index.

Though the non Newtonian lubrication world is satisfied with improved static performances journal bearing, it is still in confusion about the dynamic behaviour. Tichy and Winer [14] and Tamer [15] studied the influence of viscoelastic fluid in a squeeze film bearing. While, [16] represented a theoretical study on the stability of rotor bearing system under the influence of dilute viscoelastic lubricants.

The present work is concerned with calculation of the dynamic coefficients for journal bearing with non-Newtonian lubricants considering the power-law model.

ANALYSIS

In an η-ζ-ξ coordinates system, where η-axis is the line of centers, Figure (1), the dimensionless reaction forces due to the hydrodynamic pressure of the oil film are given by;

$$F_\eta = \iint_R P \cos\theta \, d\theta \, d\xi \tag{1}$$

$$F_\zeta = -\iint_R P \sin\theta \, d\theta \, d\xi \tag{2}$$

where R is the load-carrying region.

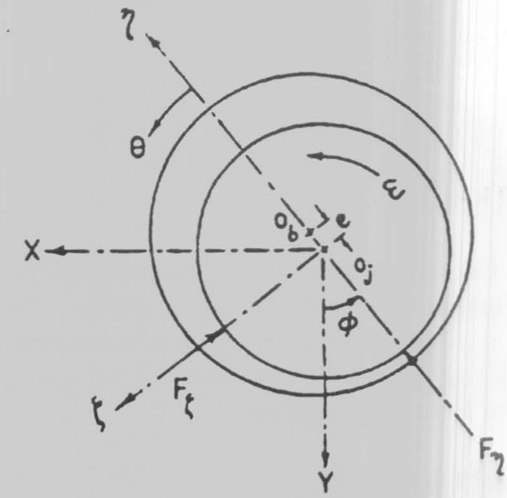

Figure 1. Coordinate system for journal bearing.

It appears that the reaction forces are function of journal center location η, ζ and its instantaneous dimensionless velocities $\dot{\eta}$, $\dot{\zeta}$ Hence, for small amplitude motion $\Delta\eta$, $\Delta\zeta$ measured from the static equilibrium position (η_o, ζ_o), a first order Taylor series expansion yields

$$F_\eta = F_{\eta_o} + K_{\eta\eta}\Delta\eta + K_{\eta\zeta}\Delta\zeta + C_{\eta\eta}\Delta\dot{\eta} + C_{\eta\zeta}\Delta\dot{\zeta} \tag{3}$$

$$F_\zeta = F_{\zeta_o} + K_{\zeta\eta}\Delta\eta + K_{\zeta\zeta}\Delta\zeta + C_{\zeta\eta}\Delta\dot{\eta} + C_{\zeta\zeta}\Delta\dot{\zeta} \tag{4}$$

where F_{η_o} , F_{ζ_o} are the static equilibrium forces and the coefficients are the partial derivatives evaluated at the equilibrium position;

$$K_{\eta\zeta} = \left(\frac{\partial F_\eta}{\partial \zeta}\right)_o , \quad C_{\eta\zeta} = \left(\frac{\partial F_\eta}{\partial \dot{\zeta}}\right)_o$$

and analogously for the remaining coefficients.

STIFFNESS COEFFICIENTS

Stiffness coefficients are dependent only on the variation of the static film reaction due to incremental displacement from the equilibrium static position. Hence, incremental forces can be calculated from equations (1), (2). Starting with the Reynolds equation governing the pressure distribution in the non-Newtonian lubricating film of a hydrodynamic journal bearing according to the power-law model given in [12,17]

$$\frac{\partial}{\partial \hat{\eta}} \frac{h^{n+2}}{n} \frac{\partial p}{\partial \hat{\eta}} + \frac{\partial}{\partial \hat{\xi}} h^{n+2} \frac{\partial P}{\partial \hat{\xi}} = 6mv^n \frac{\partial h}{\partial \hat{\eta}} \tag{5}$$

Putting $\theta = \frac{\hat{\eta}}{R}$, $\xi = \frac{\hat{\xi}}{L/2}$

$$H = \frac{h}{C}, \quad \varepsilon = \frac{e}{C}, \quad P = \frac{p}{\omega \mu_r (R/C)^2}$$

Then, the dimensionless film thickness is given as,

$$H = 1 + e \cos \theta \tag{6}$$

and equation (5) can be written in dimensionless form as,

$$\frac{\partial}{\partial\theta}\left(\frac{H^{n+2}}{n}\frac{\partial P}{\partial\theta}\right) + \left(\frac{D}{L}\right)^2 \frac{\partial}{\partial\xi}\left(H^{n+2}\frac{\partial P}{\partial\xi}\right) = 6\lambda_1 \frac{dH}{d\theta} \tag{7}$$

where $\lambda_1 = \left(\frac{\omega R}{C}\right)^{n-1}\left(\frac{m}{\mu_r}\right)$

From calculus of small variations, the variation in forces may be represented as follow; [11]

$$\begin{bmatrix} \Delta F_{\dot{\eta}} \\ \Delta F_{\zeta} \end{bmatrix} = \begin{bmatrix} \frac{\partial F_\eta}{\partial\eta} & \frac{\partial F_\eta}{\partial\zeta} \\ \frac{\partial F_\zeta}{\partial\eta} & \frac{\partial F_\zeta}{\partial\zeta} \end{bmatrix} \begin{bmatrix} \Delta\eta \\ \Delta\zeta \end{bmatrix} = \begin{bmatrix} \frac{\partial F_\eta}{\partial\epsilon} & -\frac{F_\zeta}{\epsilon} \\ \frac{\partial F_\zeta}{\partial\epsilon} & \frac{\partial F_\eta}{\epsilon} \end{bmatrix} \begin{bmatrix} \Delta\eta \\ \Delta\zeta \end{bmatrix} \tag{8}$$

then, the stiffness matrix is

$$[K_{\eta\zeta}] = \begin{bmatrix} \frac{\partial F_\eta}{\partial\epsilon} & -\frac{F_\zeta}{\epsilon} \\ \frac{\partial F_\zeta}{\partial\epsilon} & \frac{F_\eta}{\epsilon} \end{bmatrix} \tag{9}$$

and with respect to x,y coordinates is

$$[K_{xy}] = [T]\,[K_{\eta\zeta}][T]^{-1} \tag{10}$$

where $T = \begin{bmatrix} \sin\phi & \cos\phi \\ -\cos\phi & \sin\phi \end{bmatrix}$

DAMPING COEFFICIENTS

Equation (5) must be modified to compensate with velocity variation from the equilibrium position along the line of centers (radial direction) and normal to it. For normal motion the velocity component, v, in equation (5) is given by [18],

$$v = (\omega - 2\,\bar{\phi})R$$

while, for radial motion, the squeezing velocity

$$u = 2\bar{e}\cos\theta = 2\frac{\partial h}{\partial t}$$

where $\bar{\phi} = \frac{d\phi}{dt}$, $\bar{e} = \frac{de}{dt}$

Then equation (5) may take the dimensionless form

$$\frac{\partial}{\partial\theta}\left(\frac{H^{n+2}}{n}\frac{\partial P}{\partial e}\right) + \left(\frac{D}{L}\right)^2 \frac{\partial}{\partial\xi}\left(H^{n+2}\frac{\partial P}{\partial\xi}\right) = 6\lambda_1\frac{dH}{d\theta} + 6\lambda_2(\dot{\epsilon}\cos\theta)^n \tag{11}$$

where $\lambda_2 = (2\omega)^{n-1}\left(\frac{m}{\mu_r}\right)$, $\dot{\epsilon} = \frac{d\epsilon}{d\tau}$

The damping coefficients are obtained by applying incremental velocities in radial and normal directions consequently and predicting the changes in oil film forces from those at the equilibrium position.

Let, $F_{\eta\dot{\eta}}$ $F_{\zeta\dot{\eta}}$ are the forces due to applied velocity in radial direction, and $F_{\eta\dot{\zeta}}$ $F_{\zeta\dot{\zeta}}$ are the forces due to applied velocity in normal direction, then, we have;

$$\begin{bmatrix} \Delta F_\eta \\ \Delta F_\zeta \end{bmatrix} = \begin{bmatrix} \frac{\partial F_{\eta\dot{\eta}}}{\partial\dot{\eta}} & \frac{\partial F_{\zeta\dot{\eta}}}{\partial\dot{\zeta}} \\ \frac{\partial F_{\zeta\dot{\eta}}}{\partial\dot{\eta}} & \frac{\partial F_{\zeta\dot{\zeta}}}{\partial\dot{\zeta}} \end{bmatrix} \begin{bmatrix} \Delta\dot{\eta} \\ \Delta\dot{\zeta} \end{bmatrix} \tag{12}$$

So, the damping matrix is

$$[C_{\eta\zeta}] = \begin{bmatrix} \frac{\partial F_{\eta\dot{\eta}}}{\partial\dot{\eta}} & \frac{\partial F_{\zeta\dot{\eta}}}{\partial\dot{\zeta}} \\ \frac{\partial F_{\zeta\dot{\eta}}}{\partial\dot{\eta}} & \frac{\partial F_{\zeta\dot{\zeta}}}{\partial\dot{\zeta}} \end{bmatrix} \tag{13}$$

The damping coefficients in x,y directions are

$$[C_{xy}] = [T]\,[C_{\eta\zeta}]\,[T]^{-1} \tag{14}$$

STIFFNESS AND DAMPING COEFFICIENTS IN ROTOR-BEARING SYSTEM

In order to clarify the usefulness of the stiffness and damping coefficients of bearings, we consider a simple balanced rigid rotor of mass $2m_r$ supported by two symmetrical journal bearings. To study the dynamics of such systems, there are two different approaches.

The first one, motion of the center of the rotor is measured from the bearing center, which is given by the instantaneous eccentricity, e, and attitude angle, ϕ. Applying Newton's second law in the line of center direction of the bearing and normal to it, neglecting, for simplicity, friction forces, gives;

$$m_r\frac{d^2e}{dt^2} - m_r e(\frac{d\phi}{dt})^2 + f_\eta - f_e\cos\phi = 0 \qquad (15)$$

$$m_r e\frac{d^2\phi}{dt^2} + 2m_r(\frac{de}{dt})(\frac{d\phi}{dt}) - f_\zeta + f_e\sin\phi = 0 \qquad (16)$$

where f_e is a unidirectional constant force (vertical direction). Putting equations (15), (16) in dimensionless form, then

$$M\ddot{\epsilon} - M\epsilon\dot{\phi}^2 + F_\eta - F_e\cos\phi = 0 \qquad (17)$$

$$M\epsilon\ddot{\phi} + 2M\dot{\epsilon}\dot{\phi} - F_\zeta + F_e\sin\phi = 0 \qquad (18)$$

where $M = m\,\omega C^3/\mu_r LR^3$, $F_e = f_e C^2/\omega\mu_r LR^3$. Equations (17), (18) with a dynamically formulated Reynolds equation govern the dynamic behaviour of the rotor. The dynamic Reynolds equation is strongly non-linear, moreover, such equation for non-Newtonian lubricant is not established up to date.

The other approach is to consider the movements of the rotor center starting from the position of equilibrium eccentricity, that is, at $e = e_o$, x=0 and y=0, then, we have in dimensionless form;

$$M\ddot{X} + C_{xx}\dot{X} + C_{xy}\dot{Y} + K_{xx}X + K_{xy}Y = 0 \qquad (19)$$

$$M\ddot{Y} + C_{yy}\dot{Y} + C_{yx}\dot{X} + K_{yy}Y + K_{yx}X = 0 \qquad (20)$$

Equations (19), (20) are linear ordinary differential equations, hence the stiffness and damping coefficients are constant at that given equilibrium eccentricity. It is clear that the vibrational characteristics of the rotor-bearing system and its stability can be firmly determined through the dynamic coefficients of the bearing.

COMPUTATION

Equation (7) has been written in a finite difference form and has been solved by iterative method combined with an overrelaxation factor equal to 1.7 to accelerate the convergence of the pressure $P_{i,j}$ in the field i=1 to 61 (in circumferential direction) and j = 1 to 21 (in axial direction) for the journal bearing, whose specifications are;

Journal radius R = 50 mm
Radial clearance C = 0.1 mm
Viscosity consistency m = 0.0416 PaS^n
Reference viscosity μ_r = 0.0416 PaS

The iterative procedure is continued so far the relative error in computing the pressure in the next iterative procedure is more than 0.02 and is stopped when number of error points in the pressure field is less than or equal to 3 to give the final pressure distributions. Only one half of the bearing surface has been considered because of the symmetry of the pressure about the central axis plane. The components of the fluid film reaction F_η, F_ζ are evaluated using Simpson's rule. The computations have been repeated for static equilibrium eccentricity ϵ = 0.2, 0.25, ..., 0.8. Then, the derivatives of the forces F_η, F_ζ with respect to ϵ are obtained from the least square curve fitting of the ninth-order polynomials $F_\eta(\epsilon)$, $F_\zeta(\epsilon)$. Finally, the stiffness coefficients are calculated through coordinate system transformation.

During computing the damping coefficients, it was notable that, the calculated values of the pressure distribution, and the corresponding forces obtained from the numerical solution of equation (11) by applying certain velocity in normal direction through the dimensionless velocity component $V = (\omega - 2\dot{\phi})R/C\omega$ or through the component $U = 2\dot{\epsilon}\cos(\theta+\pi/2)$ are mostly identical.

Thus, the incremental forces corresponding to dimensionless velocities of -0.02, -0.01, 0.00, 0.01, 0.02 in radial direction have been obtained through $U = 2\dot{\epsilon}\cos\theta$, and in normal direction through $U = 2\dot{\epsilon}\cos(\theta+\pi/2)$.

The damping coefficients in η, ζ coordinate system are calculated by linear least square curve-fitting of the incremental forces. Then, they have been transformed to X,Y coordinate system.

The computation are carried out for different non-Newtonian fluids of power-law index, n, ranging from 0.7 to 1.2 and for the more practical bearing slenderness ratios, L/D, of 0.5, 1.0.

RESULTS AND DISCUSSION

The dimensionless stiffness coefficients are plotted versus the eccentricity ratio for different non-Newtonian lubricants specified by their flow index number, n, in figures (2,3,4,5) for L/D = 0.5 and figures (6,7,8,9) for L/D = 1.0.

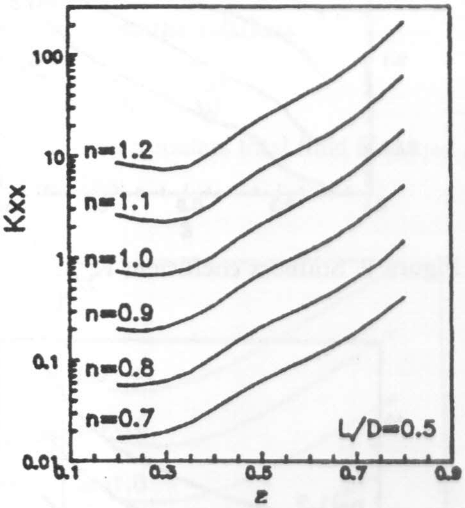

Figure 2. stiffness coefficient K_{xx}.

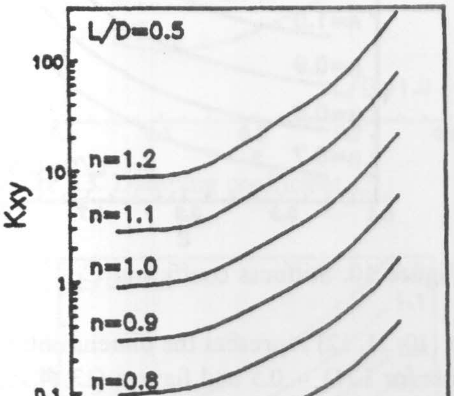

Figure 3. Stiffness coefficient K_{xy}.

When n = 1.0, the lubricant is ordinary Newtonian fluid, which is presented for the sake of comparison. For dilute lubricants, where n < 1.0, the created dynamic pressures were very low, specially for n = 0.7, consequently the values of the fluid film forces. However, the rates of change of these forces with respect to displacement, which are the stiffness coefficients, are reduced to about one fifth of that of the Newtonian fluids when the index number is reduced only by 0.1. For dilate lubricants, where n > 1.0, the dynamic pressure were sharply increased as n increased. So one can predict the high values of the stiffness coefficients, as shown also in their figures. But the rate of change of the stiffness coefficient with respect to the eccentricity ratio for all values of n is almost the same as for that of n = 1.0.

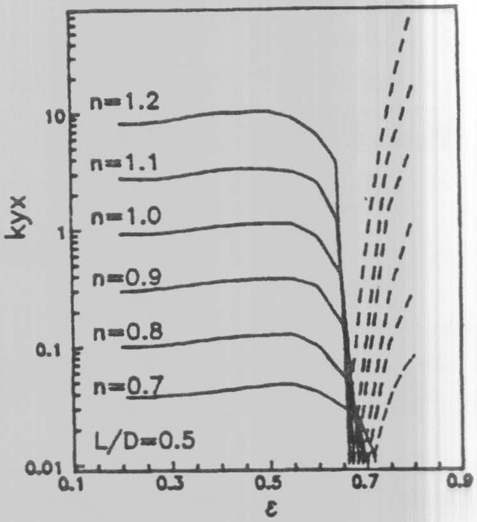

Figure 4. Stiffness coefficient K_{yx} (---- negative values).

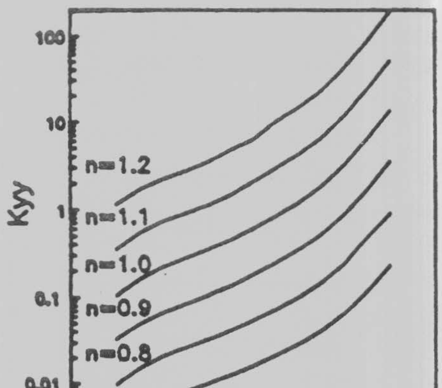

Figure 5. Stiffness coefficient K_{yy}.

Changing the ratio L/D from 0.5 to 1.0 causes, in general, a slight increase in stiffness coefficients for both Newtonian and non Newtonian lubricants. Other features of their changes remain the same.

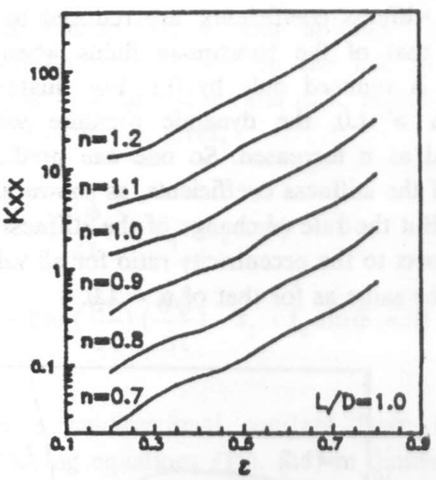

Figure 6. Stiffness coefficient K_{xx}.

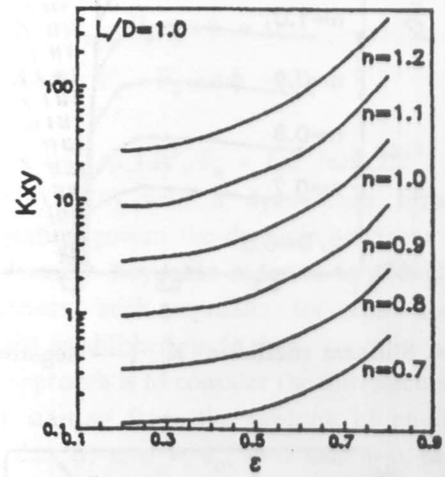

Figure 7. Stiffness coefficient K_{xy}.

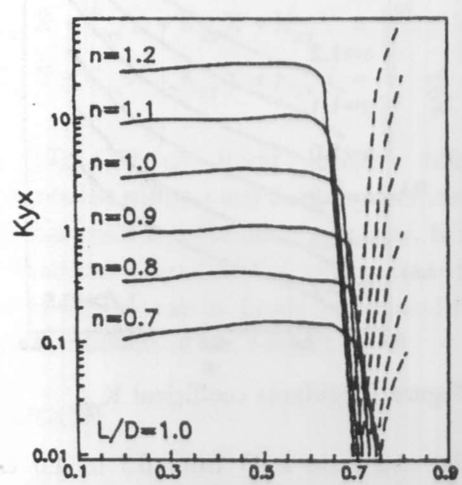

Figure 8. Stiffness coefficient K_{yx} (---- negative values).

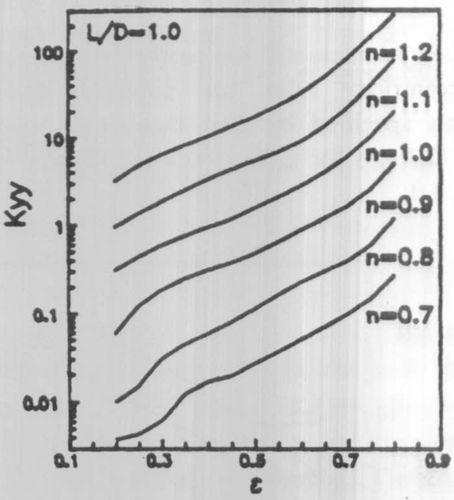

Figure 9. Stiffness coefficient K_{yy}.

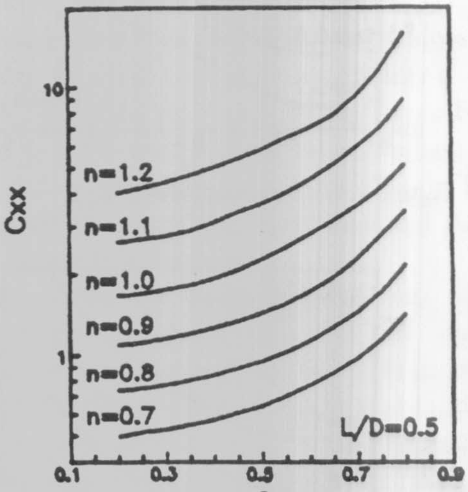

Figure 10. Stiffness coefficient C_{xx}.

Figures (10, 11, 12) represent the dimensionless damping coefficients for L/D = 0.5 and figures (13,14,15) for L/D = 1.0. The values of the cross coupling coefficients C_{xy} plotted in figures (11,14) are the mean values of C_{xy}, C_{yx}, since they have insignificant differences.

In the case of non-Newtonian lubricant, the shear stress across the oil film is some where proportional to the velocity gradient to the power n. This is equivalent to increase in viscosity, for n > 1.0, and decrease in it, for n < 1.0, as the velocity gradient increases. Thus one can conclude, that the damping coefficients increase with the increase of the flow index, n. This conclusion is proved through the above mentioned figures. Generally, these figures predict similar variation characters of the curves representing the damping coefficients to those of the stiffness coefficients. The dimensionless form of stiffness and damping coefficients presented in the recent work has been chosen to avoid, as possible, interference of curves and to perform optimum comparison. However, other dimensionless form of the dynamic coefficients as

$$\bar{K}_{xy} = \frac{kC}{W} \quad \text{and} \quad \bar{C}_{xy} = \frac{c_{xy}\,\omega\, C}{W}$$

can be obtained from the relations

$$\bar{K}_{xy} = \frac{K_{xy}}{W}, \quad \bar{C}_{xy} = \frac{C_{xy}}{W}$$

where W is the dimensionless total fluid film load at static equilibrium position .

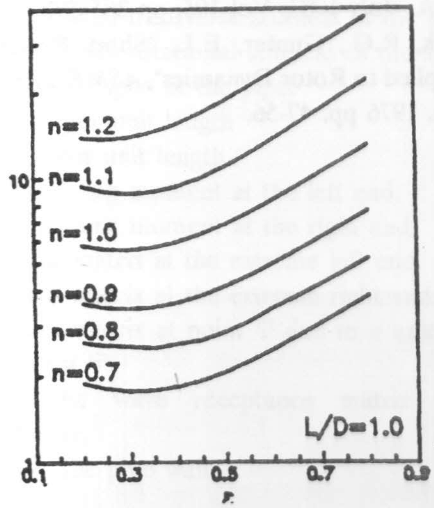

Figure 13. Damping coefficient C_{xx}.

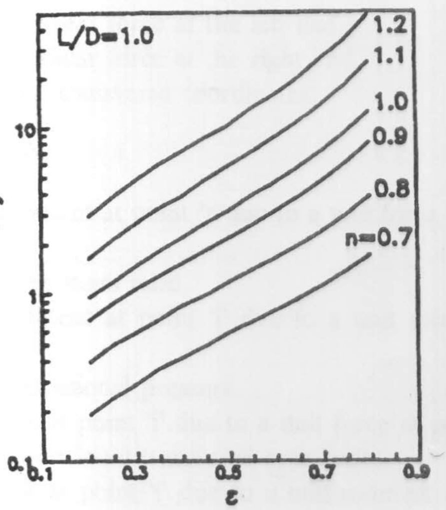

Figure 14. Damping coefficient C_{xy}.

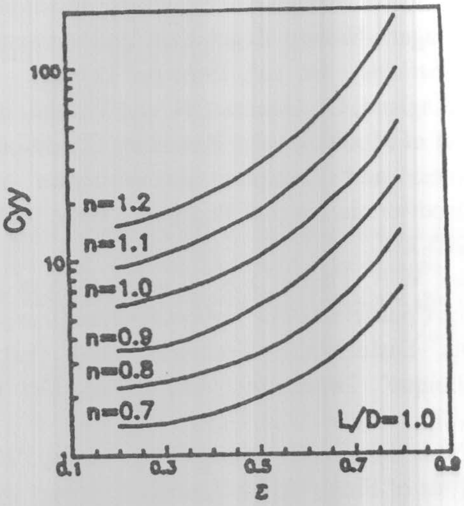

Figure 15. Damping coefficient C_{yy}.

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