NONLINEAR STATIC ANALYSIS OF STRUCTURES

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ABSTRACT

The research reported here here is concerned with the economic solution procedures for nonlinear equations of structural systems whose geometric nonlinearity is significant. The modified Newton-Raphson procedure and the BFGS (Broyden, Flectcher, Goldfrab, Shanno) method are incorporated in a computer program and examined. An accelerated procedure for the modified Newton-Raphson method using the search direction precess is also examined. Several numerical examples are given.

NOTATION

d	Vector of incremental displacements
f	Out-of-balance forces
F	Force vector corresponding to element stresses
i	Variable
I	Identity matrix
K	Stiffness matrix
R	External load vector
s	Scalar
u	Vector of displacements
y,z	Vectors
[ ]	References

INTRODUCTION

The solution procedures for the equilibrium equations are numerous. The first class of solution procedures is that which satisfies or attempts to satisfy the equation of equilibrium exactly (i.e. the out-of-balances equal zero). The methods that come under that classification are the successive approximations method, the Newton-Raphson method, and the modified Newton Raphson method. An iterative technique is necessary for convergence with the above methods. However, the method of successive approximations generally exhibits a very slow rate of convergence and a possibility of divergence for problems with significant nonlinearity. The Newton-Raphson method has proved itself to be able to converge for highly nonlinear behaviour and to control the error and estimate the rate of convergence since the iteration can be continued until a specified degree of accuracy is obtained. But, the large amount ofcomputational effort required to compute and invert, at each cycle, the coefficient matrix is considered a significant drawback of this technique. To reduce the amount of computational effort, a modified Newton-Raphson procedure can be employed where in the coefficient matrix is held constant within the load increment or possibly for several increments. The possibility of slow convergence and even divergent solution may occur with the modified Newton-Raphson method when the coefficient matrix is not updated as often as necessary.

The second class of solution procedures is that which can be classified as the incremental procedures. The basic incremental method is to divide the total load into small increments producing a corresponding incremental displacements. The coefficient matrix is calculated at the beginning of each increment. But, the equilibrium equation is generally not satisfied and the solution tends to diverge from the true deformation path. Therefore, an improved technique which includes some measures for guiding the solution back towards the true path was developed. Normally, this is done by some form of equilibrium correction, thus, these methods can be said to be self-correcting. The simplest of these techniques is to add the current force residuals to the next load increment. This load correction method corresponds to one cycle of Newton- Raphson iteration followed by a simple increment. An improvement of the above self-correcting method was suggested by Stricklin, et. al. [11] in which an amplif^Xxation factor for the residuals has been suggested.

Although self-correcting procedures do not involve equilibrium iterations, they don't guarantee that true deformation path is being found for all type of problems. Therefore some acceleration schemes for the modified Newton-Raphson method were suggested [1,2,4,5,6].

As an alternative to the Newton-Raphson method, a class of methods known as matrix update methods or Quasi-Newton methods have been developed [1,2,7,8,9].

METHOD OF SOLUTION

The conventional Newton-Raphson procedure can be written as:

$$ {}^{t}K^{(i-1)}\, d^{i} = f^{(i-1)} \qquad (1)$$

where

$$ f^{(i-1)} = R - F^{(i-1)} \qquad (2)$$

Where ${}^{t}K$ is the tangent stiffness matrix of the system, d is the incremental displacement vector, f is the out-of-balance force vector, R is the external load vector, F is the contribution of the elements internal forces vector, and i refers to the iteration number.

d^i obtained from (1) is then used to determine an improved displacement vector u^i, where

$$ u^{1} = u^{(i-1)} + d^{i} \qquad (3)$$

The Quasi-Newton method provides a secant approximation to the matrix from iteration i-1 to iteration i, that is, if

$$ d^{i} = u^{i} - u^{(i-1)} \qquad (4)$$

and

$$ \hat{f}^{i} = f^{(i-1)} - f^{i} \qquad (5)$$

then, the updated matrix should satisfy the Quasi-Newton equation

$$ K^{i}\, d^{i} = \hat{f}^{i} \qquad (6)$$

or

$$ d^{i} = (K^{-1})^{i}\, \hat{f}^{i} $$

Amongst the Quasi-Newton methods available, the BFGS method has been applied effectively in the finite element analysis [2], in which the update inverse K^{-1} can be written as

$$ (K^{-1})^{i} = (I + \hat{y}^{i}\, \hat{z}^{Ti})(K^{-1})^{(i-1)}\, (I + \hat{z}^{i}\, \hat{y}^{Ti}) \qquad (8)$$

where

$$ \hat{y}^{i} = \frac{\hat{d}^{i}}{d^{iT} \,.\, \hat{f}^{i}} \qquad (9)$$

and

$$ \hat{z}^{i} = \left(\frac{d^{i}\, \hat{f}^{i}}{d^{iT}\, K^{(i-1)}\, d^{i}} \right)^{1/2} .\ K^{(i-1)}\, d^{i} - \hat{f}^{i} \qquad (10)$$

where the superscript T means transpose.

In fact, the vector d and f of equations (7-10) are evaluated as follows:

1- Find $\bar{d} = (K^{-1})^{(i-1)}\, f^{(i-1)}$ (11)

$\bar{d}$ defines a "direction" for the actual displacement increment, and a scalar function $G(0) = d^{i} \,.\, f^{o}$ is calculated.

2- Evaluate the displacement vector $u^{i} = u^{(i-1)} + s\bar{d}$ (12) where the multiple is the value at which $G(s) \leq ETOL * G(0)$. This is usually done by evaluating the function G for a number of deodorant values of s and interpolating to some strategy. (llanos algorithm is employed to find the zero of the function G). Then

$$ d^{i} = s\, \bar{d}^{i} \qquad (13)$$

and $\hat{d}$ calculated from equation (5) is the corresponding difference.

In actual computer implementation the search direction [1,2] is performed without explicitly calculating the updated matrix of equation (8). This can be written as follows

$$ d^{i} = (I + \hat{y}^{i}\, \hat{z}^{Ti}) \ldots (I + \hat{y}^{i}\, \hat{z}^{Ti})(K^{-1})^{o}\, (I + \hat{z}^{i}\, \hat{y}^{Ti}) \ldots (I + \hat{z}^{i}\, \hat{y}^{Ti}) \,.\, \hat{f}^{i} \qquad (14)$$

Therefore, the process only involve vector and matrix multiplication and it is claimed [2] that the work remains smaller than the central back substitution as the number of factors of equation (13) is less than 15. It should be noted that the updating process is not performed if the condition number of the matrix ($I + y^{i}$ ziT) is large. The condition number C is given as [1,2]

$$ C = \left(\frac{d^{i}\, \hat{f}^{i}}{d^{iT}\, K^{(i-1)}\, d^{i}} \right)^{1/2} \qquad (15)$$

and the updating process is not performed if $C > 10^{5}$.

Several problems were saved using the modified Newton-Rapson method and the BFGS method and their results have been compared to each other. Then the search direction (equations 12-13) is used with the modified Newton-Raphson method in order to accelerate the convergence process and to make the modified method more effective.

NUMERICAL EXAMPLES

1. Large Displacement Analysis Of A Cantilever

The cantilever shown in Figure (1) was subjected to a uniformly distributed load. The finite element mesh consists of five 8-node plan stress of isotropic linear elastic material [2].

L = 10 inches. h = 1 inch.
b = 1 inch. E = 1.2 10^4 Ib/in.
μ = 0.2.

The problem was solved in 100 load steps. The B.F.G.S. method proved to be faster than the N.R. method as it solved the problem in 11102 sec. compared with 13105 sec. with N.R. Using 75 load steps, the time taken by the B.F.G.S. method was decreased to 8763 sec. while the N.R. failed to converge. Both methods failed to converge when fewer load steps are used performance in this problem giving an optimum solution time of 8018 sec. Using 50 load steps. Using fewer steps, the solution time increases again due to the increase of the number of iterations.

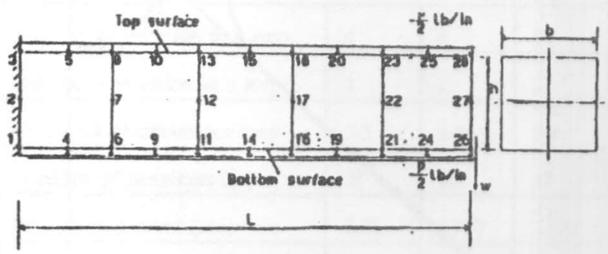

Figure 1.

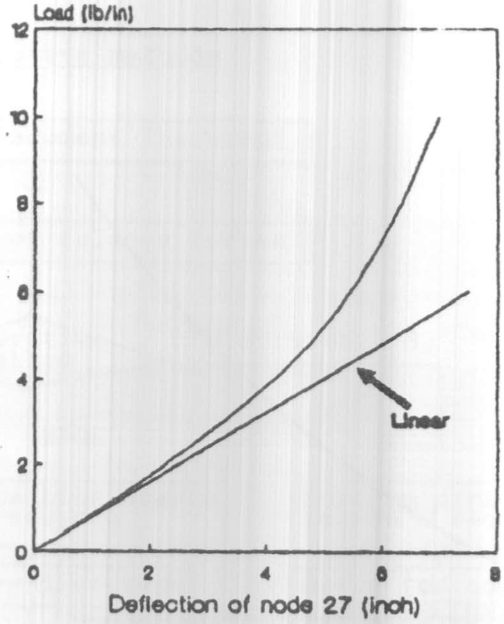

Figure 2.

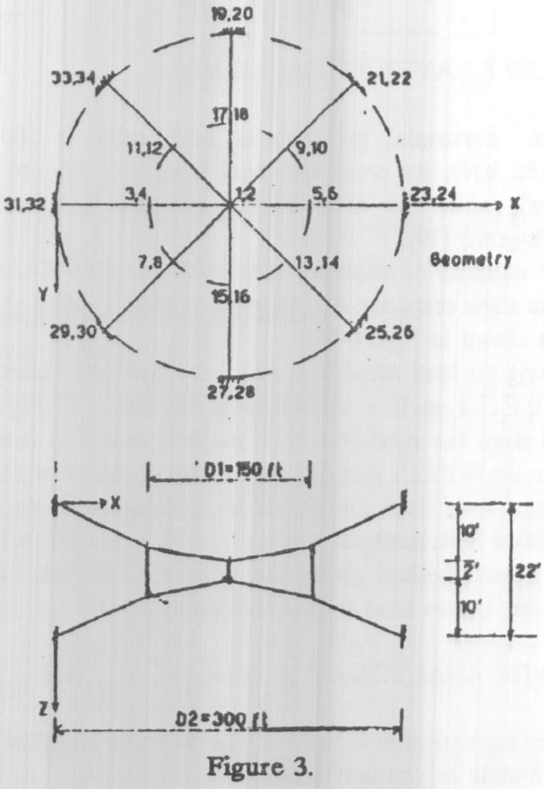

Figure 3.

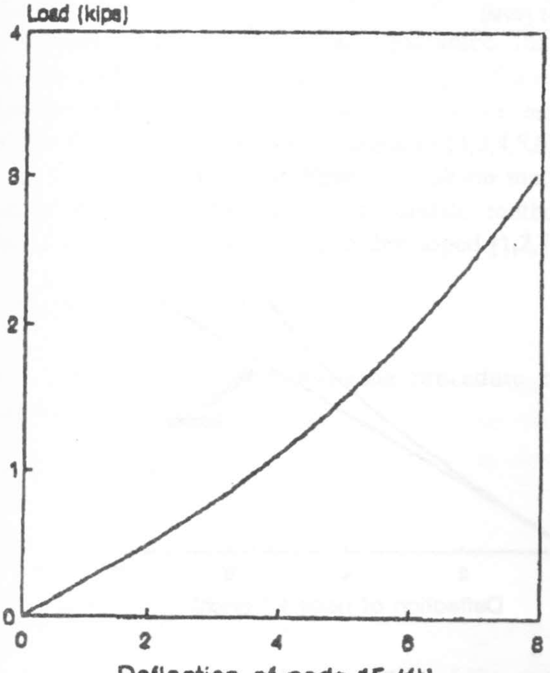

Figure 4.

DOUBLE-LAYERED CABLE NET:

The horizontal prestress in each cable is 100 kips (444.82 KN); its cross sectional area is 2 Sq. in. (12.9 sq.cm), while the area of the hangers is 0.5 sq. in. (3.23sq.cm.) [10].

the modules of elasticity E is 24000 hsi (166 Kn/mm^2).

The static response due to vertical load applied at node 15 is shown in Figure (4).

Using 10 load steps, the N.R. method was faster than the B.F.G.S. method as shown in the table. Using fewer load steps the modified N.R. method failed to converge while the B.F.G.S method solved the problem in 211 sec. Using 2 load steps and failed to converge in 1 step. The modified N.R. method with line search again proved to be the fastest method giving the optimum solution time of 197 sec. in one load step.

STATIC ANALYSIS OF A SPHERICAL CAP:

The cap shown in figure (5) was statically analyzed using ten 8-onde axisymmetric elements [3].

$\alpha = 26.67^\circ$ $\sigma_y = 24 \times 10^3$ lb/in^2

E = 10.5 × 10^6 Ib/in^2 $\in_t = 0.21 \times 810^6$ lb/in^2

$\mu = 0.3$ $\mathscr{F} = 2.45 \times 10^4$ Ib-sec/in^4

The problem was first solved in 10 load steps, then fewer steps were tried in each case. the minimum solution time was 1530 sec. with the N.R. method with line search in one step, compared with 1809 sec. with the B.F.G.S. method in 5 steps, and 3565 sec. with the N.R. method in 10 steps.

It is seen that when the modified N.R. with line search method solved the problem in one step only, the conventional N.R. method failed in less than 10 steps while the BFGS method failed in less than 5 load steps.

NONLINEAR STATIC ANALYSIS OF SIMPLY SUPPORTED BEAM

The beam is discretized into six quadratic isoparamentric elements as shown in Figure(7), subjected to its static collapse load [4].

$E = 3 \times 10^4$ kip/in2

$\mu = 0.3$

The powerful effect of the line search technique in reducing the number of load steps without causing divergence appears clearly in this problem. When the problem couldn't be solved in less than 16 steps using the N.R. method and the B.F.G.S method as divergence

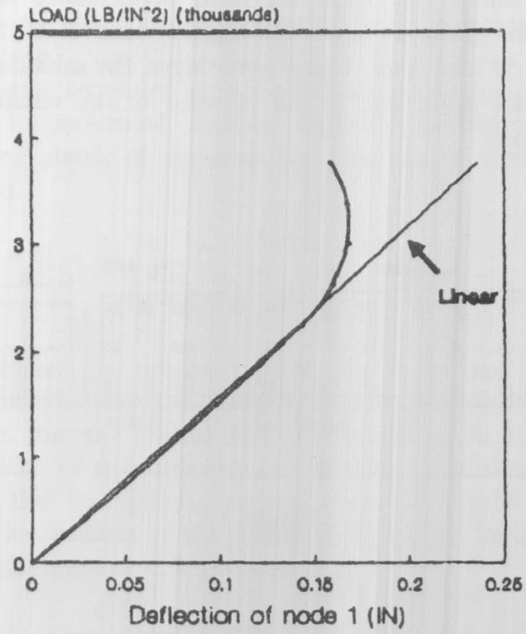

Figure 5.

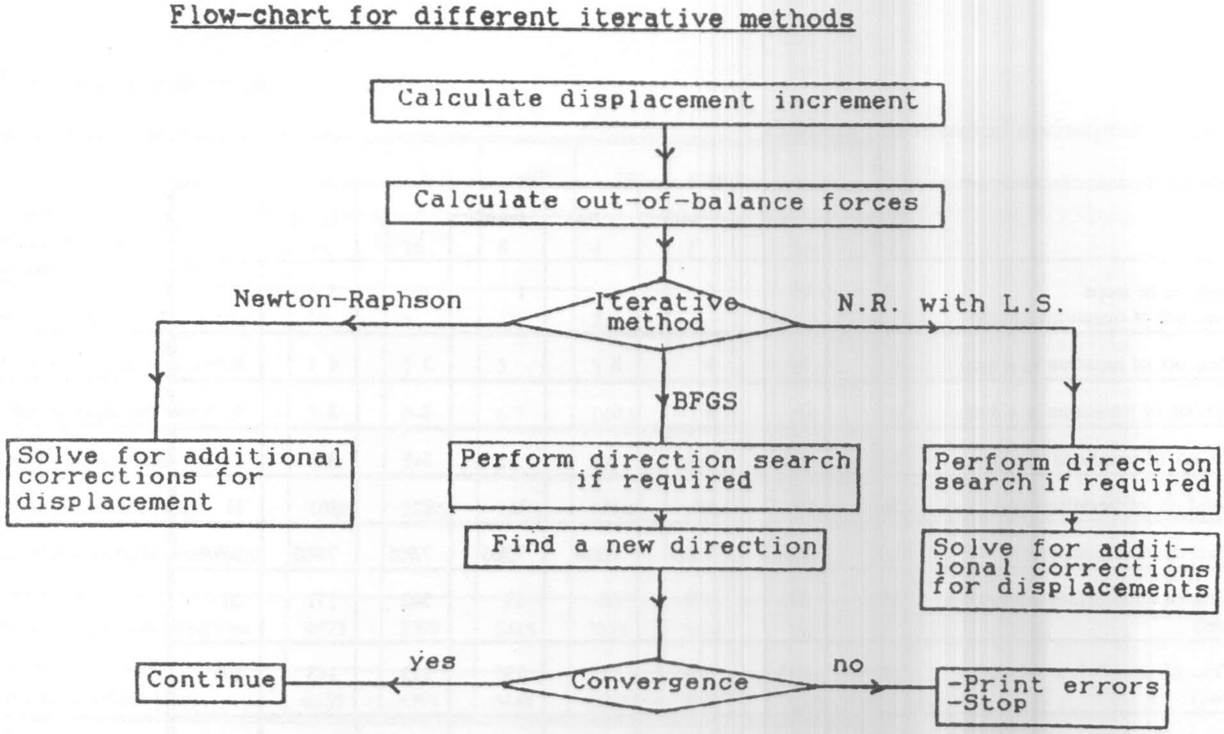

Figure 6.

Table 1. Comparison between the methods

	N.R	N.R	with	line	search	B. .F	G. S	
Number of steps Max. no. of iterations in step	10	10	5	2	1	10	5	2
Max. no. of iteration in a step	4	4	5	8		3	4	7
Min on. of iteration in a step	2	2	3	6		2	2	4
Av. no. of iterations per step	2-3	2-3	3-4	7		2-3	3-4	5-6
Total no. pf iterations (in)	28	28	19	14	16	22	16	11
Final displacement (sec)	7.91	7.912	7.909	7.914	7.905	7.905	7.905	7.908
Time of equilibrium interations (sec)	342	341	172	68	33	342	171	68
Time of equilibrium iterations (sec)	209	215	144	117	136	213	143	105
Total solution time (sec)	666	677	384	223	197	680	385	211

Table 2. Comparison between the methods

	N.R.	N.R.	with	line	search	B. .F	G. S	
Number of steps Max. no. of iterations in step	10	10	5	2	1	10	5	2
Max. no. of iteration in a step	4	4	5	8		3	4	7
min. on of iterations in a step	2	2	3	6		2	2	4
Av. no. of iterations per step	2-3	2-3	3-4	7		2-3	3-4	5-6
Total no. pf iterations (in)	28	28	19	14	16	22	16	11
Final displacement (sec)	7.91	7.912	7.909	7.914	7.905	7.905	7.905	7.908
Time of equilibrium interations (sec)	342	341	172	68	33	342	171	68
Time of equilibrium iterations (sec)	209	215	144	117	136	213	143	105
Total solution time (sec)	666	677	384	223	197	680	385	211

Table 3. Comparison between the methods

	N.R.	N.R.	with	line	search	B. .F	G. S
Number of steps Max. no. of iterations in step	10	10	5	2	1	10	5
Max. no. of iteration in a step	20	7	13	26		4	5
min. on of iterations in a step	2	2	3	7		2	3
Av. no. of iterations per step	4-5	3-4	5-6	16-17		2-3	3-4
Total no. pf iterations (in)	45	32	92	33	31	27	19
Final displacement (sec)	0.145	0.146	0.146	0.146	0.147	0.146	0.146
Time of equilibrium interations (sec)	119	120	60	24	12	121	57
Time of equilibrium iterations (sec)	1948	1556	1404	1849	1367	1381	1014
Total solution time (sec)	3565	3140	2196	1530	2975	2975	1809

Table 4. Comparison between the methods

	N.R.	N.R.	with	line	search	
Number of steps Max. no. of iterations in step	16	16	8	4	1	16
Max. no. of iteration in a step	53	5	13	27		4
Min on. of iteration in a step	1	1	3	3		1
Av. no. of iterations per step	7-8	3-4	6-7	14-15		2-3
Total no. pf iterations (in)	126	54	51	57	57	42
Final displacement (sec)	2.786	2.786	2.786	2.786	2.774	2.786
Time of equilibrium interations (sec)	77	78	40	20	4	80
Time of equilibrium iterations (sec)	4473	2399	2445	3115	3594	1968
Total solution time (sec)	6458	4393	3448	3618	3721	3975

Table 5. Comparison between the methods

	N.R.	N.R.	with	line	search	B.	F.	G.	S.
Number of steps Max. no. of iterations in step	15	15	7	3	1	15	7	3	1
Max. no. of iteration in a step	18	4	6	8		4	4	6	
Min on. of iteration in a step	1	1	1	2		1	12		
Av. no. of iterations per step	2-3	1-2	2-3	4-5		1-2	2	3-4	
Total no. pf iterations (in)	38	22	17	14	15	21	14	118	
Final displacement (sec)	4.28	4.28	4.278	4.282	4.137	4.281	4.279	4.277	4.287
Time of equilibrium interations (sec)	180	181	85	36	12	180	85	36	11
Time of equilibrium iterations (sec)	249	177	133	116	131	187	117	102	94
Total solution time (sec)	607	542	312	200	168	558	300	191	131

5- HYPERBOLIC NET :

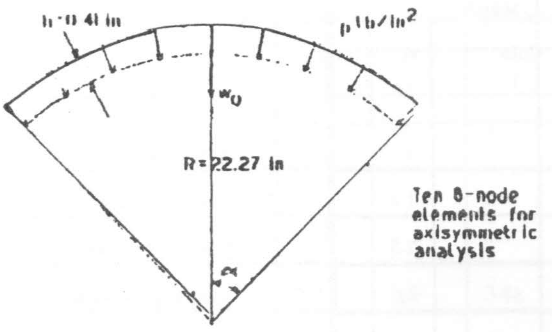

Fig. (5).

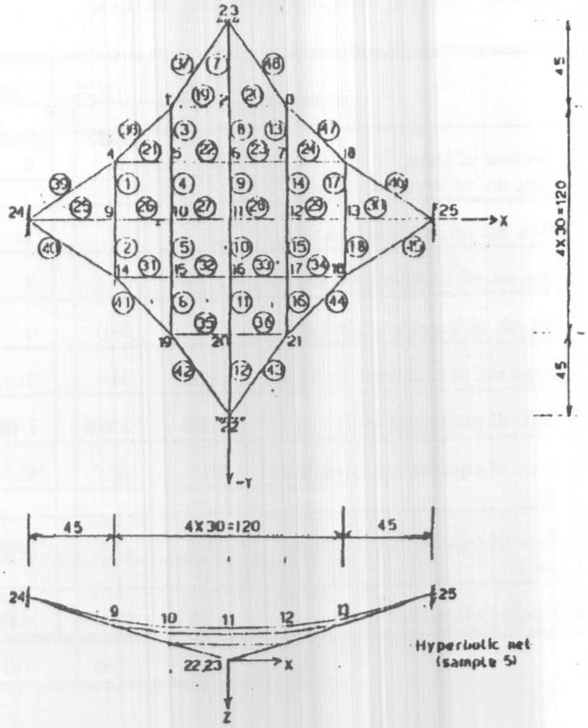

Figure 9.

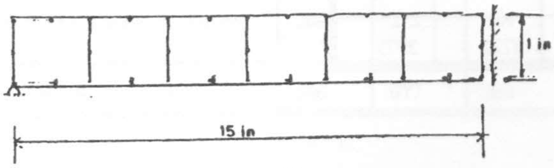

Figure 7.

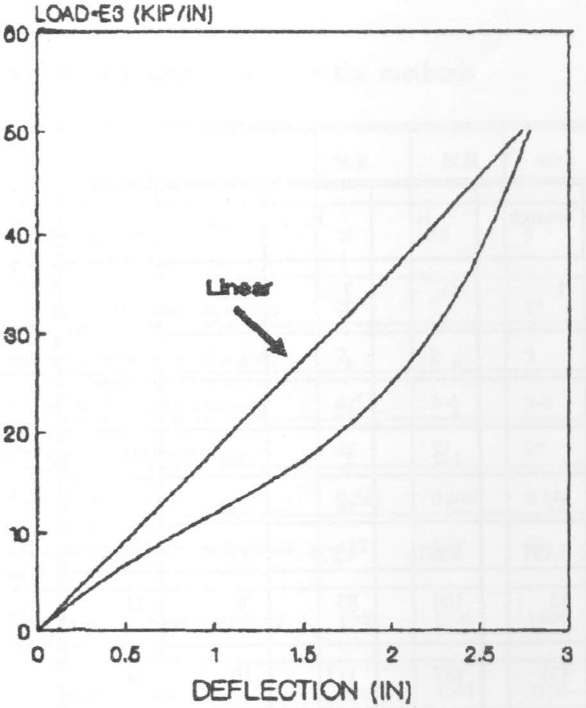

Figure 8.

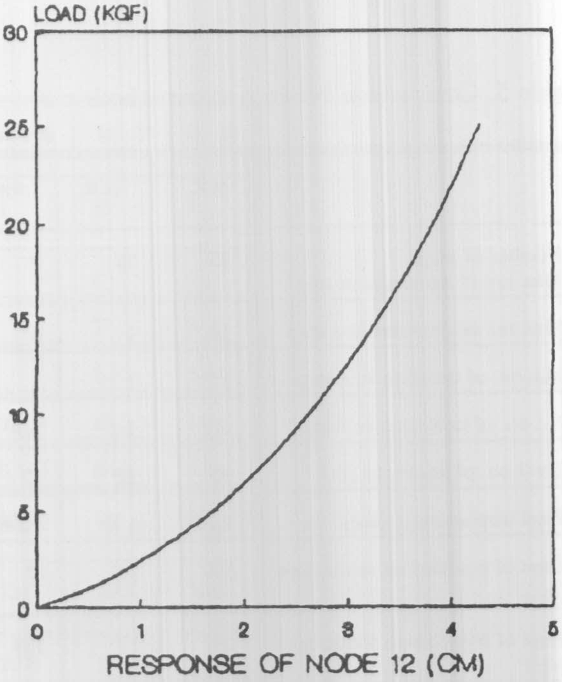

Figure 10.

occurred, it was solved in only one step with the N.R. method with line search, but the optimum solution time was obtained using 8 load steps as the average number of iterations increases significantly using fewer load steps.

The hyperbolic net shown in Figure (9) and is its necessary data were given by Morris [10]. The EA values are: EA = 15500 kgf (152 KN) for the interior cables and 40300 kgf (395Kn) for the boundary cables.

When solving the problem in 15 steps, the N.R. method with line search achieved the minimum solution time of 542 sec., but the optimum solution time of the problem was obtained using the B.F.G.S. method. It should be noted that using the line search process, almost 75% of the solution time of the original N.R. method was reduced.

CONCLUSION

The standard Modified Newton-Raphson method needed more time and load steps than the BFGS method in almost all the problems solved.

Using the proposed technique of introducing the line-search process into the Modified Newton-Raphson method, the solution time and the number of load steps needed were reduced to the minimum, and although, in some problems, the solution time of the BFGS method was less than that of the proposed method when using the same number of load steps, it was noticed that the BFGS method failed to converge when using fewer load steps while the proposed method did converge.

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