ON THE SOLUTION OF SOME ASYMMETRIC OSCILLATIONS

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ABSTRACT

An approximate solution of the nonlinear differential equation; $\ddot{u}-\alpha^2u^2 = \beta$, is considered where α and β are constants. The approximate solution obtained is shown to correspond to the exact one for a particular set of initial conditions. An example in the form of $\ddot{x}+3x^2-6x+2 = 0$ which can be transformed to the above equation is studied.

INTRODUCTION

Many physical phenomena in engineering and technology are modelled by second-order nonlinear differential equations [1-3]. In order to analyse the behaviour of the physical situations, the solutions of the differential equations, which are in general difficult to obtain in nonlinear cases, are needed. However, there are methods to obtain approximate solutions of such nonlinear differential equation [1-3]. For example consider the cases when the differential equation takes the form,

$$\ddot{x}+f(x) = 0, \quad (\dot{\ } = d/dt) \tag{1}$$

Such problems have been extensively studied, for example as in [4-6], where f(x) is an odd function. Meanwhile the case, where f(x) is neither even nor odd, (as in the case of asymmetric oscillations), Howard [7] and Bernard [8(i), (ii)] have changed f(x) into two fictive oscillators, each of them is symmetric, and by this way they obtained the periodic time of oscillation for this case.

Peters [9] and Usher [10] have obtained some approximate solution to the differential equation $\ddot{x}+\omega^2x = -\alpha x^2$, subject to the initial conditions $x(0) = A$, $\dot{x}(0) = 0$. But their approximate solution is not reduced to the exact one namely, $x(t) = -\omega^2/\alpha$ for the special case when $A=-\omega^2/\alpha$. Such a difficulty has been eliminated by Shidfar and Sadeghi [11] using different approach.

Thandapani [6] used the method of Shidfar and Sedeghi [11] to obtain an approximate solution of the following nonlinear equations [13],

$$\ddot{x}+\alpha^2x^2 = \beta^2x^4 \tag{2}$$

$$\ddot{x}+\alpha^2x^2 = -\beta^2x^3 \tag{3}$$

with initial conditions

$$x(0) = A \text{ and } \dot{x}(0) = 0 \tag{4}$$

In the present paper, we use the method described in [12] and [13] to obtain the approximate solution of the inharmonic motion equation (Bernard [8(i)]), given by

$$\ddot{x}+3x^2-6x+2 = 0 \tag{5-a}$$

with initial conditions

$$x(0) = A \text{ and } \dot{x}(0) = 0 \tag{5-b}$$

which has the form of equation (1), where $f(x) = 3x^2-6x+2$. Using a special transformation which will be described later, equation (5) can be rewritten in the following form;

$$\ddot{x}+\omega^2[x+\epsilon G(x)] = 0 \tag{6}$$

where ϵ is a small parameter.

To arrive at equation (6), one can apply the method illustrated in [8(i), (ii)], by finding the roots of f(x), one gets $x_0 = 1 + 1/\sqrt{3}$, then translating f(x) to $f(x+x_0)$, equation (5) reduces to the from of equation (6) as follows:

$$\ddot{x}+2\sqrt{3}\,[x+(\sqrt{3}/2)x^2] = 0 \tag{7}$$

The solution of equation (7) has been studied using approximate methods in [8] and using exact methods in [7-8].

In this paper, we are interested to find the series solution of equation (5) which, can be written in the form

$$\ddot{u}-\alpha^2u^2 = \beta \tag{8-a}$$

$$u(0) = u_0, \; u_{(0)} = 0 \tag{8-b}$$

where α and β are constants, then by substituting

$$x = -u+1 \tag{9}$$

into equation (5), we obtain

$$\ddot{u}-3u^2 = -1 \tag{10-a}$$

with

$$u(0) = -A+1, \; \dot{u}(0) = 0 \tag{10-b}$$

SERIES SOLUTION

It is required to solve the nonlinear differential equations of the type given in equation (2), (3) or (8). Since the type given by equation (8) has not been previously considered, we concentrate our investigation on this equation. We seek solution of the form

$$u(t) = \sum_{n=0}^{\infty} C_n \sin^n \alpha t; \text{ n integer} \tag{11}$$

where the coefficients C_n, n = 0, 1, 2, ..., are constants to be determined.

Substituting the series (11) into (8-b), we get $C_0 = U_0$. It can be shown that $C_1 = 0$ from equations (8-b), and (11). Substituting (11) into the left hand side of equation (8-a) and equating the sum of the coefficients of each $\sin^n \alpha t$, n = 0, 1, 2, ..., to zero, we obtain the set of coefficients C_n, n = 0 , 1, 2, ...

It is interesting to note that the series solution (11) is absolutely convergent for all t and consequently it converges for all values of t. This conclusion is given in [12].

EXAMPLE

In this section, we determine the series solution of equation (5-a) with initial conditions (5-b), but in its form of equations (10-a) and (10-b). We seek a series solution for equation (10-a) in the form

$$U(t) = C_0+C_1\sin\sqrt{3}\,t+C_2\sin^2\sqrt{3}\,t + C_3\sin^3\sqrt{3}\,t+C_4\sin^4\sqrt{3}\,t+... \tag{12}$$

where C_0, C_1, C_2, C_3, C_4, ..., are constants to be determined.

From equation (12), we can get the following;

$$\dot{U}(t) = C_1\sqrt{3}\cos\sqrt{3}\,t+C_2.2\sqrt{3}\sin\sqrt{3}\,t\cos\sqrt{3}\,t + C_3.3\sqrt{3}\sin^2\sqrt{3}\,t\cos\sqrt{3}\,t + C_4.4\sqrt{3}\sin^3\sqrt{3}\,t\cos\sqrt{3}\,t+... \tag{13}$$

$$\ddot{U}(t) = -3C_1\sin\sqrt{3}\,t+6C_2(1-2\sin^2\sqrt{3}\,t) + 9C_3\sin\sqrt{3}\,t(2-3\sin^2\sqrt{3}\,t) + 12C_4\sin^2\sqrt{3}\,t(3-4\sin^2\sqrt{3}\,t)+... \tag{14}$$

$$U^2(t) = C_0^2+(C_0C_1+C_1C_0)\sin\sqrt{3}\,t+(C_0C_2+C_1^2+C_2C_0). \sin^2\sqrt{3}\,t+(C_0C_3+C_1C_2+C_2C_1+C_3C_0)\sin^3\sqrt{3}\,t + (C_0C_4+C_1C_3+C_2^2+C_3C_1 + C_4C_0)\sin^4\sqrt{3}\,t+... \tag{15}$$

Substituting equations (14) and (15) into (10-a) and using equation (13) in the initial conditions (10-b), we conclude that

$$C_1 = C_3 = 0; C_0 = -A+1; C_2 = (3A^2-6A+2)/A; \quad C_4 = (3A^2-6A+2)(3-A)/36. \tag{16}$$

The required solution of equation (5) can be obtained though equations (9), (12) and (16), and it has the following from;

$$x(t) = A-[(3A^2-6+2)/6]\sin^2\sqrt{3}\,t + [(3A^2-6A+2)(A-3)/36]\sin\sqrt{3}\,t+... \tag{17}$$

The solution for special cases A = 1, 2, 3, 4, 5, 6, and $1+1/\sqrt{3}$ has the following forms.

$$x(t) = 1+(1/6)\sin^2\sqrt{3}\,t+(1/18)\sin^4\sqrt{3}\,t+... \tag{18}$$

$$x(t) = 2-(1/3)\sin^2\sqrt{3}\,t-(1/18)\sin^4\sqrt{3}\,t+... \tag{19}$$

$$x(t) = 3-(11/6)\sin^2\sqrt{3}\,t+... \tag{20}$$

$$x(t) = 4-(13/3)\sin^2\sqrt{3}\,t+(13/18)\sin^4\sqrt{3}\,t\;... \tag{21}$$

$$x(t) = 5-(47/6)\sin^2\sqrt{3}\,t+(47/18)\sin^4\sqrt{3}\,t+... \tag{22}$$

$$x(t) = 6-(74/6)\sin^2\sqrt{3}\,t+(74/12)\sin^4\sqrt{3}\,t+... \tag{23}$$

$$x(t) = 1+1/\sqrt{3} \tag{24}$$

We observe that the coefficients C_2, C_4, C_6, ... vanish for $A = 1+1/\sqrt{3}$. Therefore $x(t) = 1+1/\sqrt{3}$ for $A = 1+1/\sqrt{3}$ and $x(t) = 1-1/\sqrt{3}$ for $A = 1-1/\sqrt{3}$.

Hence the solution of the differential equation (5) corresponds to the exact one for these particular initial conditions, and that is in full agreement with the work of Bernard [8(i), (ii)].

Approximate solutions and results for different values of initial conditions, equations (18-24), are give in Figures (1), (2), ... (7).

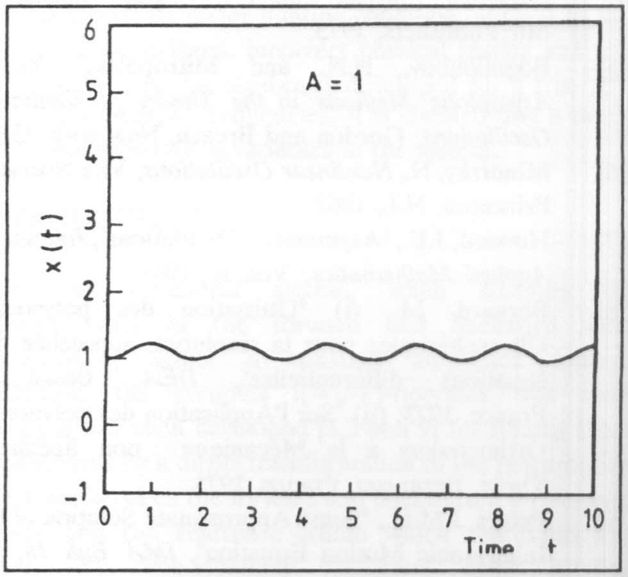


Figure 1.

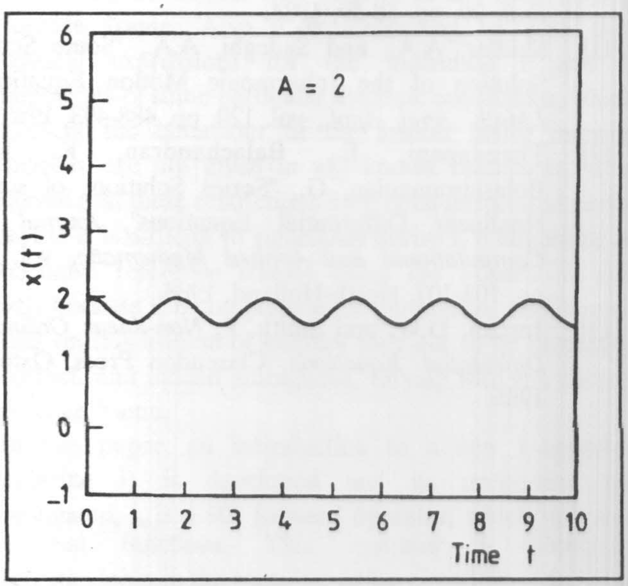
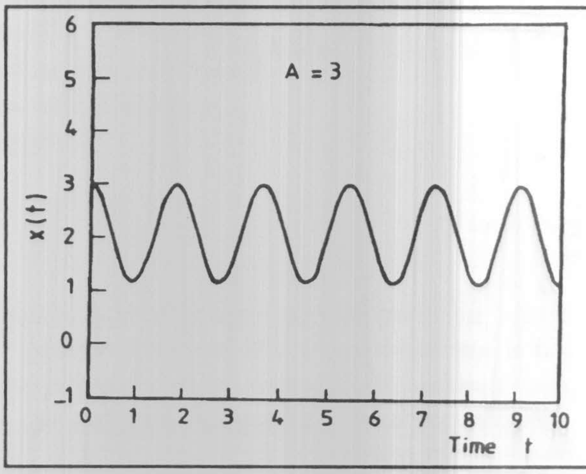


Figure 2.



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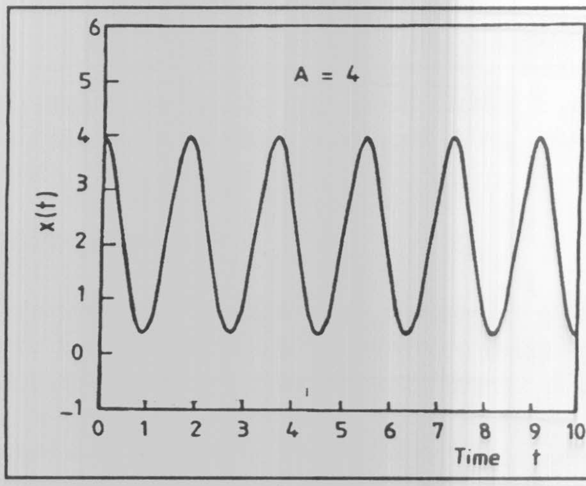

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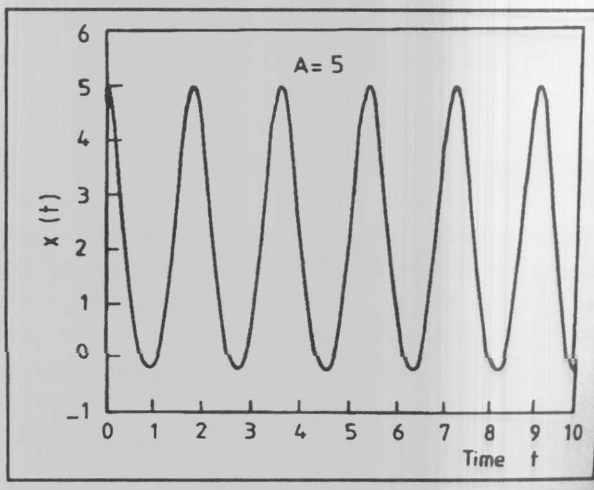

Figure 5.

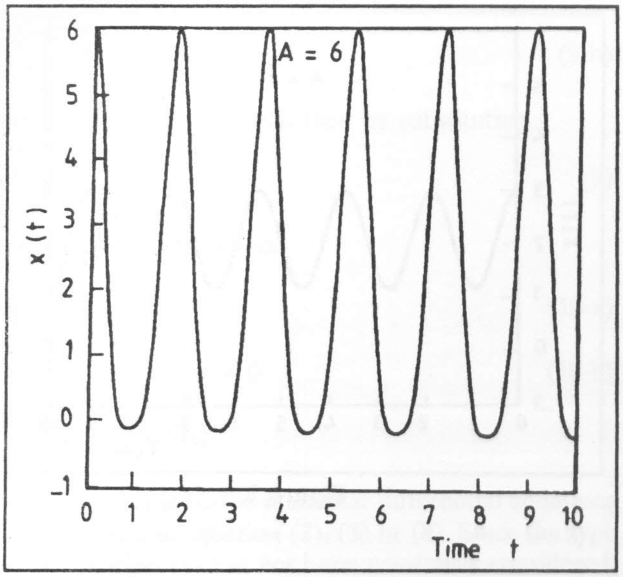

Figure 6.

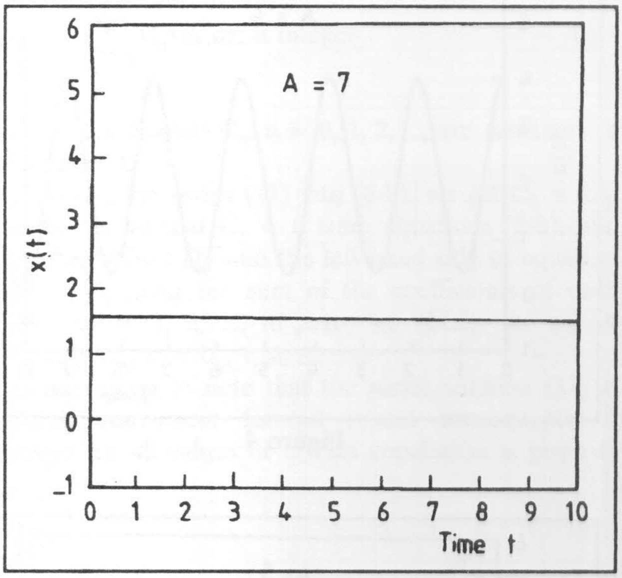

Figure 7.

CONCLUSION

From the above investigation, one concludes that the solution of the nonlinear differential equation (5) is periodic (symmetric), although the function $f(x) = 3x^2-6x+2$ in equation (5-a) is asymmetric.

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