ON MONOTONICITY OF THE POWER FUNCTION OF THE LRT AGAINST RESTRICTED ALTERNATIVES FOR BIVARIATE NORMAL DISTRIBUTION

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ABSTRACT

This paper addresses the problem of testing simple hypothesis about the mean of a bivariate normal distribution with identity covariance matrix against one sided alternative. It is shown that the power function of the LRT satisfies certain monotonicity property. In addition, the null distribution of the LRT statistic is derived.

1. INTRODUCTION

Consider a bivariate normal vector (X,Y) with mean $\theta=(\theta_1, \theta_2)$ and identity covariance matrix. Assume V to be convex closed cone in $\mathbb{R}^2$ with vertex at (0,0). It is required to test,

$$H_0:(\theta_1,\theta_2) = (0,0) \text{ versus } H_1:(\theta_1,\omega_2) \in V\backslash\{(0,0)\}. \quad (1)$$

Many authors have considered this type of testing problems. Bartholomew (1959a) considered a test for homogeneity of means for ordered alternatives. He devoted his work to deriving the LRT and finding its null distribution. Due to the difficulty in solving certain recurrence relation, the distribution is not completely determined. After this problem had been solved by Miles (1959), Bartholomew (1959b) extended his previous work and gave some further properties of the null distribution. Perlman (1969) has considered the testing problem which is discussed by Bartholomew (1959a, 1959b), but under thee assumption that the covariance matrix is completely unknown. Kudô (1961) and independently, Nüesch (1966) have considered a testing problem for which the alternative space is the non-negative quadrant.

In this paper we will show that the power function of the LRT for the testing problem H_0 versus H_1 described in equation (1) has the following monotonicity property: Assume that the alternative (θ_1,θ_2) can be represented in polar coordinates as (Δ,γ). It is shown that for fixed Δ the power function is increasing in γ for $\gamma \langle \gamma_0$, where γ_0 is determined by the cone V. Although Bartholomew (1961a) has motivated this property by some numerical examples for certain cones, the result was not established theoretically. He could only conjecture that this property holds for any convex closed cone. Al-Rawwash (1988) used this property to prove certain conjecture about the LRT. In Section 2, we present the derivation of the LRT. Section 3 is devoted to proving the monotonicity property, as well as to deriving the null distribution of the LRT statistic in the bivariate case.

2. DERIVATION OF THE LRT

Let V be the closed convex cone in $\mathbb{R}^2$ with vertex at (0,0), which is given in polar coordinates by

$$V = \{(r,\beta): r\geq 0,\ 0 \leq\beta\leq\beta^*\} \quad (2)$$

where β^* satisfies either $0\leq\beta^*\leq\pi$ or $\beta^*=2\pi$. We need one of these two conditions to be satisfied by β^* so that V is a convex cone. The testing problem which we are concerned with is of the form (1), with V having the form (2).

For the case $\beta^*=2\pi$, it is known that the LRT for such testing problem is the usual χ^2-test by which H_0 is accepted for small values of x^2+y^2. We will follows Bartholomew (1959a) to obtain the LRT for the case $0\leq\beta^*\leq\pi$. Partition the space $\mathbb{R}^2$ into four regions given by:

$$V = \{(r,\beta): r\geq 0,\ 0\leq\beta\leq\beta^*\}$$

$$V_1 = \{(r,\beta): r\geq 0,\ \beta^*<\beta\leq\beta^*+\pi/2\}$$

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$V_2 = \{(r,\beta): r \geq 0,\ \beta^* + \pi/2 \leq \beta \leq 3\pi/2\}$

$V_3 = \{(r,\beta): r \geq 0,\ 3\pi/2 < \beta < \pi/2\}$

In fact each region constitutes a cone in $\mathbb{R}^2$. These cones are illustrated graphically in Figure (1).

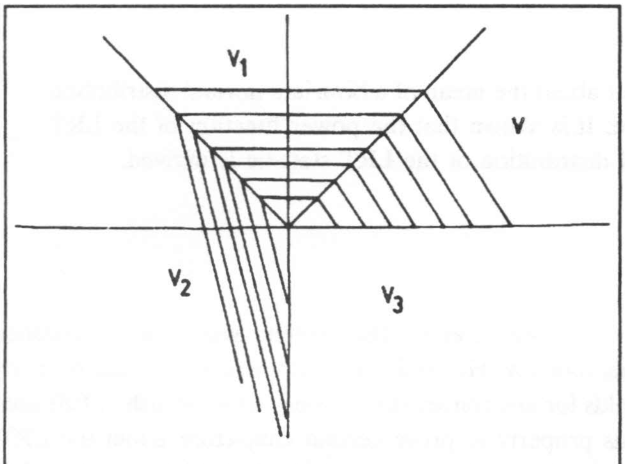

Figure 1. The four cones that make a partition for $\mathbb{R}^2$.

Bartholomew (1959a) has shown that the LRT is given by

$$\phi = \begin{cases} 1 & \bar{\chi}^2 > k \\ 0 & \bar{\chi}^2 \leq k \end{cases} \tag{3}$$

where

$$\bar{\chi}^2 = x^2 + y^2 - (x-\hat{\theta}_1)^2 - (y-\hat{\theta}_2)^2 \tag{4}$$

and $(\hat{\theta}_1, \hat{\theta}_2)$ is the MLE of (θ_1, θ_2) restricted to V.

It can be shown that [See Kudô (1963) and Bartholomew (1961a, 1961b)] the MLE of (θ_1, θ_2) restricted to V is given by

$$(\hat{\theta}_1, \hat{\theta}_2) = \begin{cases} (x,y) & \text{if } (x,y)\epsilon V \\ (0,0) & \text{if } (x,y)\epsilon V_2 \\ (x,0) & \text{if } (x,y)\epsilon V_3 \\ (c,bc) & \text{if } (x,y)\epsilon V_1 \end{cases} \tag{5}$$

where $b = \tan\beta^*$, $c = (x+by)/(1+b^2)$. From this it can be shown that

$$\bar{\chi}^2 = \begin{cases} x^2+y^2 & \text{if } (x,y)\epsilon V \\ 0 & \text{if } (x,y)\epsilon V_2 \\ x^2 & \text{if } (x,y)\epsilon V_3 \\ c(x+by)^2 & \text{if } (x,y)\epsilon V_1 \end{cases} \tag{6}$$

The acceptance region of the LRT is depicted in Figure (2).

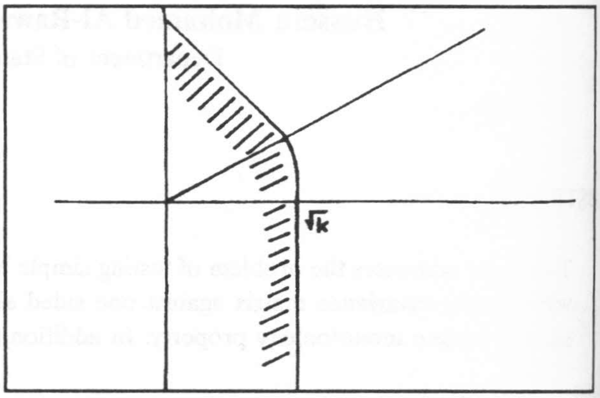

Figure 2. The acceptance region of the LRT.

3. THE MONOTONICITY OF THE POWER FUNCTION OF THE LRT

In this section we shall explicitly obtained the power function of the LRT derived in Section 2. In addition, we will prove that the power function is symmetric about $\beta^*/2$. Further, we show that the power function satisfies some monotonicity property. First, we obtained the power function of the test ϕ_V. To proceed we may adopt the following notations.

$$Z(w) = (1/\sqrt{2\pi})\, e^{-w^2/2}, \quad Q(w) = \int_w^{\infty} Z(t)\, dt.$$

For $(\theta_1, \theta_2)\epsilon V$ let (Δ, γ) be given by

$$\Delta^2 = \theta_1^2 + \theta_2^2,\ \gamma = \tan^{-1}(\theta_1/\theta_2) \tag{7}$$

It is obvious that (Δ, γ) is the polar transformation of (θ_1, θ_2). Also for $(x,y)\epsilon\mathbb{R}^2$, let (r,ζ) be its polar transformation.

We divide the rejection region of ϕ_V into three parts as illustrated in Figure (3).

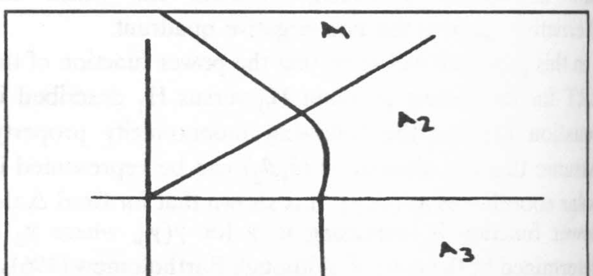

Figure 3. The three parts of the rejection region.

Notice that, A_1, A_2 and A_3 may be represented as follows:

$$A_1 = \{(x,y): x+by>\sqrt{k(1+b^2)} \text{ and } y>bx\} \text{ if } b\geq 0$$

or

$$A_1 = \{(x,y): x+by<\sqrt{k(1+b^2)} \text{ and } y<bx\} \text{ if } b<0$$

$$A_2 = \{(x,y): (x^2+y^2)>k \text{ and } (x,y)\epsilon V\}$$

$$A_3 = \{(x,y): x>\sqrt{k},\ y<0\}$$

Also notice that

$$\{(x,y): \bar{\chi}^2>k\} = A_1 \cup A_2 \cup A_3 \tag{8}$$

where A_1, A_2 and A_3 are mutually exclusive sets. The power function is given by

$$\beta(\theta_1,\theta_2) = E_{(\theta_1,\theta_2)}\,\phi_V = P_{(\theta_1,\theta_2)}(A_1\cup A_2\cup A_3)$$
$$= P_{(\theta_1,\theta_2)}(A_1)+P_{(\theta_1,\theta_2)}(A_2)+P_{(\theta_1,\theta_2)}(A_3)$$

where $P_{(\theta_1,\theta_2)}$ is the probability measure induced by a bivariate normal distribution with mean vector (θ_1,θ_2) and identity covariance matrix. We will drop the subscript (θ_1,θ_2) for conveniency. Assume that

$$\mu(\Delta,\gamma) = \beta(\Delta\cos\gamma,\Delta\sin\gamma) \tag{10}$$

where Δ and γ are related to θ_1 and θ_2 as in (7).

The probability of each of A_i, $i=1,2,3$ under P, will be evaluated successively as follows: To evaluate $P(A_1)$ we have to make a rotation of angle β^*; where the new coordinates (x',y') are related to (x,y) by the following relations;

$$x' = x\cos\beta^*+y\sin\beta^* \text{ and } y' = y\cos\beta^*-x\sin\beta^*$$

In view of the above transformation (x',y') is distributed as bivariate normal distribution with mean (μ_x,μ_y) and identity covariance matrix, where

$$\mu_x = \Delta\cos(\beta^*-\gamma) \text{ and } \mu_y = \Delta\sin(\gamma-\beta^*) = -\Delta\sin(\beta^*-\gamma)$$

In view of this, we have

$$P(A_1) = Q(\sqrt{k} - \Delta\cos(\beta^*-\gamma))Q(\Delta\sin(\beta^*-\gamma))$$

If we let

$$G(t) = Q(\sqrt{k} - \Delta\cos t)Q(\Delta\sin t) \tag{11}$$

then,

$$P(A_1) = G(\beta^*-\gamma) \tag{12}$$

Now we proceed to evaluate $P(A_3)$. By the structure of A_3 we have,

$$P(A_3) = \int_{-\infty}^{0}\int_{\sqrt{k}}^{\infty} Z(x-\theta_1)\, Z(y-\theta_2)\, dy\, dx$$
$$= [1-Q(-\theta_2)][Q(\sqrt{k}-\theta_1)] = Q(-\theta_2)Q(\sqrt{k}-\theta_1)$$

Using the polar coordinate of (θ_1,θ_2) as given in (7) we have

$$P(A_3) = Q(\Delta\sin\gamma)Q(\sqrt{k} - \Delta\cos\gamma) = G(\gamma) \tag{13}$$

where G is given by (11). Finally we evaluate $P(A_2)$:

$$P(A_2) = \int\int_{A_2} (Z(x-\theta_1)\, Z(y-\theta_2)\, dy\, dx$$

By using the polar representation (Δ,γ) of (γ_1,θ_2) and (r,ζ) of (x,y) we get

$$P(A_2) = \int_0^{\beta^*}\int_{\sqrt{k}}^{\infty} (1/2\pi)e^{-\frac{1}{2}(r^2+\Delta^2)+r\Delta\cos(\zeta-\gamma)}r\,dr\,d\zeta$$
$$= [(e^{-\Delta^2/2})/2\pi]\int_0^{\beta^*} H(k,\Delta\cos(\zeta-\gamma))\, d\zeta$$
$$= [(e^{-\Delta^2/2})/2\pi]\int_{-\gamma}^{\beta^*-\gamma} H(k,\Delta\cos\zeta)\, d\zeta$$
$$\equiv H^*(\gamma), \text{ say,} \tag{14}$$

where

$$H(k,t) = \int_{\sqrt{k}}^{\infty} e^{-\frac{1}{2}r^2+rt}dr = \sqrt{2\pi}\;e^{t^2/2}Q(\sqrt{k}-t)$$

In view of (12), (13) and (14), we have

$$\eta(\Delta,\gamma) = G(\beta^*-\gamma)+G(\gamma)+H^*(\gamma) \tag{15}$$

Now we can state our first result.

<u>Theorem 1</u>: For the case that $\beta^*\leq\pi$ and for fixed Δ, the power function $\eta(\Delta,\gamma)$ is symmetric in γ about $\beta^*/2$, i.e.

$$\eta(\Delta,\beta^*-\gamma) = \eta(\Delta,\gamma) \quad \text{if } \gamma\leq\beta^*/2 \tag{16}$$

Proof: We will prove that the two functions $H^*(\gamma)$ and $G^*(\gamma)$ are symmetric in γ about $\beta^*/2$, where H^* is given by (14) and G^* is given by

$$G^*(\gamma) = G(\beta^*-\gamma)+G(\gamma),$$

and G is given by (11). It is obvious that G^* is symmetric. We have to show that H^* is symmetric too. Notice that

$$H^*(\beta^*-\gamma) = [(e^{-\Delta^2/2})/2\pi] \int_{\gamma-\beta^*}^{\gamma} H(k,\Delta\cos\zeta)\, d\zeta$$

Change the variable of integration from ζ into $-\zeta$ we get

$$H^*(\beta^*-\gamma) = -[(e^{-\Delta^2/2})/2\pi] \int_{-\gamma+\beta^*}^{-\gamma} H(k,\Delta\cos(-\zeta))d\zeta$$

$$= [(e^{-\Delta^2/2})/2\pi] \int_{-\gamma}^{\beta^*-\gamma} H(k,\Delta\cos\zeta)\, d\zeta$$

$$= H^*(\gamma).$$

This completes the proof of the theorem■

We will prove a similar result for the case $\beta^*=2\pi$.

Theorem 2: For the case that $\beta^*=2\pi$, and for fixed Δ, the power function $\eta(\Delta,\gamma)$ is constant in γ for $0\le\gamma\le 2\pi$.

Proof: Notice that for the case $\beta^*=2\pi$ the LRT may be reduced to the usual χ^2-test which is given by

$$\phi_V(x,y) = \begin{cases} 0 & \text{if } x^2+y^2<k \\ 1 & \text{if } x^2+y^2\ge k \end{cases} \qquad (17)$$

hence the power function will be

$$\eta(\Delta,\gamma) = P(X^2+Y^2\ge k)$$

$$= \int_0^{2\pi}\int_{\sqrt{k}}^{\infty} (1/2\pi)e^{-\frac{1}{2}(r^2+\Delta^2)+r\Delta\cos(\zeta-\gamma)}\, rdrd\zeta$$

$$=[(e^{-\Delta^2/2})/2\pi]\int_{\sqrt{k}}^{\infty} e^{-r^2/2}\left[\int_0^{2\pi} e^{r\Delta\cos(\zeta-\gamma)}d\zeta\right] rdr \quad (18)$$

Now

$$\int_0^{2\pi} e^{r\Delta\cos(\zeta-\gamma)}d\zeta = \int_{\gamma}^{2\pi+\gamma} e^{r\Delta\cos\zeta}d\zeta = \int_0^{2\pi} e^{r\Delta\cos\zeta}d\zeta \quad (19)$$

The last equality holds because the function $\cos\zeta$ is a periodic function of period 2π. In view of (19), equation (18) becomes,

$$\eta(\Delta,\gamma) = [(e^{-\Delta^2/2})/2\pi]\int_{\sqrt{k}}^{\infty} e^{-r^2/2}\left[\int_0^{2\pi} e^{r\Delta\cos\zeta}d\zeta\right] rdr$$

which is free of γ. This completes the proof■

Based on some numerical computations, Bartholomew (1961) noticed that for fixed Δ, the power function $\eta(\Delta,\gamma)$ has a bell shape in γ, for the special case $\beta^*=\pi/3$. He conjectured that this is true for all values of $\beta^*\le\pi$. In the following theorem we will prove this conjecture.

Theorem 3: For $\beta^*\le\pi$ and for fixed Δ, the power function $\eta(\Delta,\gamma)$ is an increasing function in γ for $0\le\gamma\le\beta^*/2$ and a decreasing function for $\beta^*/2\le\gamma\le\beta^*$.

Proof: By the symmetry on η about $\beta^*/2$, it is enough to show that η is increasing for $0\le\gamma\le\beta^*/2$.

Assume B is the acceptance region of the LRT. Therefore

$$\eta(\Delta,\gamma) = 1-\int\int_B (1/2\pi)e^{-\frac{1}{2}(x^2+y^2)-\frac{1}{2}\Delta^2+\Delta x\cos\gamma+\Delta y\sin\gamma}dxdy$$

Take the derivative of η with respect to γ we get

$$\frac{\partial\eta(\Delta,\gamma)}{\partial\gamma} = -\int\int_B (1/2\pi)e^{-\frac{1}{2}(x^2+y^2)-\frac{1}{2}\Delta^2+\Delta x\cos\gamma+\Delta y\sin\gamma} \times [-\Delta x\sin\gamma+\Delta y\cos\gamma]\, dxdy.$$

Now make a transformation of (x,y) into (x′,y′), where $x'=x\cos\gamma+y\sin\gamma$ and $y'=y\cos\gamma-x\sin\gamma$.

It can be shown that (x′,y′) is distributed as a bivariate normal distribution with mean vector $(\Delta,0)$ and identity covariance matrix. Therefore,

$$\frac{\partial\eta(\Delta,\gamma)}{\partial\gamma} = -\int\int_{B'} M(x',y')dx'dy'$$

where

$$M(x',y') = (1/2\pi)\, \Delta y' e^{y'^2+(x'-\Delta)^2}$$

and B′ is the image of B in x′y′-plane. The region B′ is depicted in Figure (4). It can be seen from the figure that B′ can be divided into three disjoint regions B_1, B_2 and B_3 as illustrated in the same figure.

Because B_1 is a mirror image of B_2 and because of the structure of M(x′,y′) we have

$$\int\int_{B_1} M(x',y')dx'dy' = -\int\int_{B_2} M(x',y')dx'dy'.$$

Also, because $y'<0$ in B_3

$$\int_{B_3}\int M(x',y')dx'dy' < 0.$$

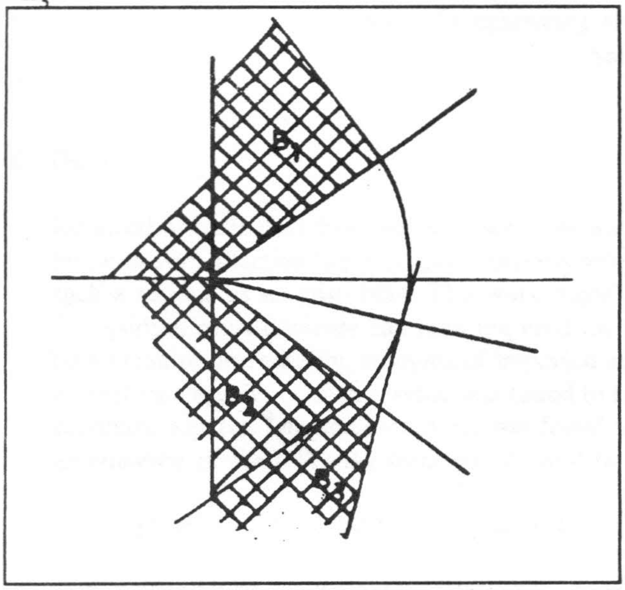

Figure 4. The set B′ and its partitions B_1, B_2 and B_3.

Thus

$$\frac{\partial\eta(\Delta,\gamma)}{\partial\gamma} > 0 \text{ for } 0<\gamma<\beta^*/2.$$

This completes the proof■

In the last part of this paper we will derive the null distribution of the statistics $\bar{\chi}^2$. Let

$$F(k) = 1-P_{(0,0)}(\bar{\chi}^2>k)$$

We know that

$$P_{(0,0)}(\bar{\chi}^2>k) = P_{(0,0)}(A_1)+P_{(0,0)}(A_2)+P_{(0,0)}(A_3)$$

where A_1, A_2 and A_3 are given before. It can be easily seen that

$$P_{(0,0)}(A_1) = \tfrac{1}{2}\,Q(\sqrt{k}) = P_{(0,0)}(A_3)$$

and

$$P_{(0,0)}(A_2) = (\beta^*/2\pi)\; e^{-k/2}.$$

Hence

$$F(k) = 1-Q(\sqrt{k}) - (\beta^*/2\pi)\; e^{-k/2}.$$

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