AN INTRODUCTION TO A NEW PHYSICAL OPERATOR, k-OPERATOR

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ABSTRACT

This paper is presented according to the mathematical difficulties in treating some physical problems, related to any dynamic system such as, for example in the electric machine dynamics. The use of i-operator in this respect makes some difficulties in understanding the physical behaviour of the forward and backward wave components under hunting condition, applied to the system. This may cause a rigorous lack between the work of some authors, incorrect physical results and ambiguity. To avoid the obscurities of existing mathematical treatments, the algebra is developed for a true 90° forward time operator that works with real variables and with multiple frequencies. It is claimed that a much clearer understanding of all matters relating to forward f and backward b variables in the system.

INTRODUCTION

In any dynamical system, when studying the characteristics of the forward and backward wave components, under a constant amplitude hunting condition, the complex $i(=\sqrt{-1})$-operator was used. Analysis has been developed [1,3 and 9] for finding these components by a direct transformation of the perturbation variables between the forward and backward f, b reference frame and the reference frame which represents the system by time-invariant coefficient under hunting condition.

However, it should be appreciated that the f and b waves can possibly be viewed with respect to any time-varying reference frame. Also it is interesting to note that analytical expressions for the individual f and b components of some particular dynamic coefficients, which represent the behaviour of the system under hunting condition, are not given in any known references. It is believed that these restrictions have been introduced partly because it is difficult to rigorously derive f, b equations in alternative reference frames using the i-operator, and partly because f, b components of the hunting coefficients based on this avenue of enquiry turn out to be physically incorrect, and indeed ambiguous, varying with the chosen reference frame.

In this paper, an introduction to a new k-operator (replacing i) is developed and its properties are investigated, k is a 90° forward operator, which operates on real functions. This operator is physically understandable in the transformation equations, where i is not, and it enables the process of deriving f and b equations to be rigorously understood in any reference frame.

THE k-OPERATOR

In developing f, b equations for perturbation variables under hunting condition, a transformation matrix with i notation has been used [2,4]. This matrix has been defined to be available to transform either (or both) the perturbation or the steady-state variables between hunting equations and f,b components (or vice versa by the use of inverse). It is stressed that the quantity i in this matrix is allowed to operate upon both the perturbation and the steady-state variables, although f and b components of the steady-state variables, of zero frequency, are physically meaningless. In case, the steady-state variables have no time dependence and it is impossible to say whether they represent a forward wave, or a backward wave, or some combination of the two.

The i notation used earlier, viewed as a forward operator for a phaser variable, is strictly available to operate only on the perturbation variables expressed in the time-invariant reference frame, in which they contain only one frequency (perturbation frequency β). f and b components can then be derived from the perturbation equations, with $p=i\beta$, where $p=d/dt$ is the differential operator. But the i operator is inadequate for developing those components from system equations expressed in any time-varying reference frame which contains multiple frequencies in its perturbation variables.

To overcome this problem, a new operator to replace i is proposed, denoted by k, which is available for the perturbation variables (only of any dynamic system expressed in any reference frame. This operator proves to be very convenient for use in a generalized transformation matrix **K**, as it can operate on trigonometric functions of one or more frequencies. It is mathematically defined as:

$$k = \frac{p}{|\omega|} = \frac{\text{differential time operator}}{\text{frequency modules}}$$

COMPARISON BETWEEN i AND k

The operator k has similar characteristics to i, but i is for functions while k operates only on real functions. Consideration shows that i is 90° forward for positive frequency but 90° backward with negative frequency (an important point which is not generally appreciated [1,8], whereas k is 90° forward with either positive or negative frequency. A comparison between i and k is given in the following examples (1) and (2), assuming cosωt and sinωt are two trigonometric functions of frequency ω, where $\omega > 0$.

The operator works on a complex function and the instantaneous values before and after the operation are conventionally obtained from the real parts. Thus the expressions before the colon are in complex form, and after the colon, the implied results for instantaneous variables (by taking the real parts) are given, before and after the i operation.

$$ie^{i\omega t}: \cos\omega t \rightarrow -\sin\omega t, \quad ie^{-i\omega t}: \cos\omega t \rightarrow \sin\omega t$$

$$i(ie^{i\omega t}): -\sin\omega t \rightarrow -\cos\omega t \quad \text{or } \sin\omega t \rightarrow \cos\omega t$$

$$i(ie^{-i\omega t}): \sin\omega t \rightarrow -\cos\omega t \qquad (1)$$

Similarly for the k-operator we have:

$$k(\cos\omega t): \cos\omega t \rightarrow -\sin\omega t, \quad k(\cos(-\omega t)): \cos\omega t \rightarrow -\sin\omega t$$

$$k(\sin\omega t): \sin\omega t \rightarrow \cos\omega t, \quad k(\sin(-\omega t): \sin\omega t \rightarrow \cos\omega t \qquad (2)$$

The difference between the two equation sets of (1) and (2) lies firstly in their handling of negative frequency; it is seen that i with a negative frequency becomes effectively a <u>backward</u> 90° operator, whereas k with negative frequency remains effectively a <u>forward</u> 90° operator. Also, of course, i must operate on a complex operand in the form $e^{i\omega t}$, whereas k is equipped to operate on real variables which naturally occur in transforming spatial reference frames.

It follows that with the new operator k, with any operand whose frequency is the difference of two (positive) frequencies, the transformed result depends on which frequency is the larger. Whilst that property is mathematically awkward (though it can be handled quite readily in practice), it does express the true situation. It reveals in transformation between spatial reference frames, for example, how a particular wave which is forward-moving relative to one frame may become backward-moving relative to another.

PROPERTIES OF k-OPERATOR

This section explores the previous property and tabulates important algebraic results for the k-operator in this connection (property 7). Also, rules are developed by which k can be used to operate on product terms of several different frequencies (property 8), avoiding the need to first multiply out these terms into single frequency components.

k has the following properties, by considering operation on the general, simple trigonometric function: $X = a\cos(\pm\omega t + \phi)$, where $\omega > 0$.

1) k=90° forward, $-k = 1/k$ = 90° backward. Thus:

$$k(\cos(\omega t+\phi)) = \cos(\omega(t+\pi/2\omega)+\phi) = -\sin(\omega t+\phi)$$

$$-k(\cos(\omega t+\phi)) = \cos(\omega(t-\pi/2\omega)+\phi) = \sin(\omega t+\phi)$$

$$k(\sin(-\omega t+\phi)) = \sin(-\omega(t+\pi/2\omega)+\phi) = -\cos(\omega t-\phi)$$

$$-k(\sin(-\omega t+\phi)) = \sin(-\omega(t-\pi/2\omega)+\phi) = -\cos(\omega t-\phi)$$

$$k(\cos(-\omega t+\phi)) = -\sin(\omega t-\phi)\text{-}k(\cos(-\omega t+\phi)) = \sin(\omega t-\phi)$$

$$k(\sin(\omega t+\phi)) = \cos(\omega t+\phi)\text{-}k(\sin(\omega t+\phi)) = -\cos(\omega t+\phi)$$

Clearly, operator k always causes the right-hand side to lead (and -k causes it to lag) the operand of the left-hand side by 90° in time, regardless of the sign of frequency. (note that i, with complex variables, does not possess this property).

2) $k^2=-1$, $k^3=-k$, $k^4=1$, $k+(-k)=0$ and $k(-k)=1$

3) If $p=d/dt$, then $pk=kp$, i.e. k and p are commutative.

4) $k(cX)=ck(X)$, where c is constant,

5) If $X=a\cos\omega t$ and $Y=b\sin\beta t$ are two different simple functions each of one frequency but the respective frequencies $\pm\omega$ and $\pm\beta$ may be different, then $k(X+Y)=k(X)+k(Y)$.

6) If $n=\tan\theta$, then $(1+nk)/\sqrt{1+n^2}$ is the forward operator by angle θ.
(if $\theta=0$, the operator is 1 ($n=0$); if $\theta=90°$, the operator is k ($n=\infty$)).

7) If $\omega>0$, $\beta>0$ and t is the time, we have some important properties for difference frequencies, which depend on the relative magnitude of ω and β.

$$k(\sin(\omega-\beta)t)=\frac{\omega-\beta}{|\omega-\beta|}\cos(\omega-\beta)t=\begin{cases}\cos(\omega-\beta)t & \text{for } \omega>\beta\\ -\cos(\omega-\beta)t & \text{for } \omega<\beta\end{cases}$$

$$k(\sin(\beta-\omega)t)=\frac{\beta-\omega}{|\beta-\omega|}\cos(\beta-\omega)t=\begin{cases}-\cos(\omega-\beta)t & \text{for } \omega>\beta\\ \cos(\omega-\beta)t & \text{for } \omega<\beta\end{cases}$$

$$k(\cos(\omega-\beta)t)=\frac{\omega-\beta}{|\omega-\beta|}(-\sin(\omega-\beta)t)=\begin{cases}-\sin(\omega-\beta)t & \text{for } \omega>\beta\\ \sin(\omega-\beta)t & \text{for } \omega<\beta\end{cases}$$

$$k(\cos(\beta-\omega)t)=\frac{\beta-\omega}{|\beta-\omega|}(-\sin(\beta-\omega)t)=\begin{cases}-\sin(\omega-\beta)t & \text{for } \omega>\beta\\ \sin(\omega-\beta)t & \text{for } \omega<\beta\end{cases}$$

8) $$k(XY) = =\begin{cases}Yk(X) & \text{for } \omega>\beta\\ Xk(Y) & \text{for } \omega<\beta\end{cases}$$

which is proved as follows:

$$\begin{aligned}k(XY)&=k(ab\cos\omega t\sin\beta t)=abk(\sin(\beta+\omega)t+\sin(\beta-\omega)t)/2\\ &=ab(\cos(\beta+\omega)t+\cos(\beta-\omega)t)/2\\ &=\cos\omega t\cos\beta t \text{ for } \omega<\beta\text{, equals } Xk(Y)\\ \text{or} \quad &=abk(\sin(\omega+\beta)t-\sin(\omega-\beta)t)/2\\ &=ab(\cos(\omega+\beta)t-\cos(\omega-\beta)t)/2\\ &=-\sin\omega t\sin\beta t \text{ for } \omega>\beta\text{, equals } Yk(X).\end{aligned}$$

Note, in 7 and 8 above, the results are indeterminate for $\omega=\beta$. In this case the difference $\beta-\omega$ is zero and the function turns out to be constant (steady-state with zero frequency). k does not operate on constant.

TRANSFORMATION MATRIX

The transformation matrix **K** is defined for two perturbation variables, coinciding on two axes at right angle, and for f, b components [6,7 and 9], expressed in any reference frame, as follows:

$$\mathbf{K}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -k & k\end{bmatrix}\text{; the inverse is}$$

$$\mathbf{K}^{-1}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & k\\ 1 & -k\end{bmatrix} \qquad (3)$$

By introducing the matrix $\mathbf{K}^-$ which is **K** with all k terms reversed in sign (similar to the complex conjugate), that is:

$$\mathbf{K}^-=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ k & -k\end{bmatrix}$$

the following properties are satisfied.

$$(\mathbf{K}^-)^{-1}=(\mathbf{K}^{-1})^-=\mathbf{K}^T;\ (\mathbf{K}^{T-})^{-1}=\mathbf{K};\ \mathbf{K}^{T-}=\mathbf{K}^{-1};\ (\mathbf{K}^T)^{-1}=\mathbf{K}^- \quad (4)$$

where T denotes transposition.

The use of the transformation matrix is important to study f, b wave components in any dynamic system. This matter is pursued in the next sections.

POWER TRANSFORMATION

It has long been appreciated [5] that the electromechanical conversion of energy (or power) is a concept of fundamental physical importance in regard to dynamic studies based on perturbation variables. With all systems excited by cyclic variables, it is important to distinguish between instantaneous, time-average and oscillatory power flows. If for example the power is the product between two variables in the system (u,v) input, and the instantaneous values of these two components represent sinusoidal oscillation at frequency ω, then the complex form of u and v are:

$u = \hat{u}e^{i\omega t}, \; v = \hat{v}e^{i(\omega t - \phi)},$

where $\hat{u}$ and $\hat{v}$ are the peak values, and ϕ is the phase angle between the phasor u and phasor v. Therefore

the time-average power = $\mathrm{Re}(u^* v)/2 = (\hat{u}\hat{v}\cos\phi)/2$,

where * indicates the complex conjugate.
Oscillatory power is given by, Re(uv)/2, that is

Oscill. power = $\frac{1}{2}\mathrm{Re}(\hat{u}\hat{v}e^{i\omega t}e^{i(\omega t-\phi)}) = \frac{1}{2}\hat{u}\hat{v}\cos(2\omega t-\phi)$,

and the total instantaneous power by:

$$\mathrm{Re}(u)\mathrm{Re}(v) = \hat{u}\hat{v}\cos\omega t\cos(\omega t-\phi) = \tfrac{1}{2}\hat{u}\hat{v}(\cos\phi + \cos(2\omega t-\phi)).$$

Clearly, in this case the instantaneous power = time-average power + the oscillatory power.

The same results (principally for time-average power) for complex variables are reflected in equated in equations for real variables using the k-operator. This is appreciated from general vectors in complex form U and V, where $(U^{T*})V = U^T(V^*)$, and introduce an important general principle for k-operator with real functions X and Y. It is immediately seen that: $((kX)Y)_{av} = (X(-kY))_{av}$, because advancing the phase of X relative to Y is equivalent to retarding the phase Y relative to X, so far as time-average effects are concerned. This can be generalized for any f(k) as follows:

$$((f(k)X)Y)_{av} = (X(f(-k)Y))_{av} \qquad (5)$$

Equation (5) is the necessary result for exploring power-invariant transformations with the k-operator. It shows the way in which the k-operator can be detached from X and attached to Y, or vice versa, if only average power is concerned. But it should be appreciated that the instantaneous values of the products in (5) do not obey the same law. This means that transformations which maintain time-average power invariant do not in general maintain instantaneous power invariant; this statement applied equally to complex phaser transformations, as to real transformations with the k-operator, though it is frequently not appreciated.

Power-invariant Transformation

We can express power relationships between two general variables in vector form **X** and **Y** in both two axis frame and f, b frame as follows:

P = the time-average power $(\mathbf{X}^T\mathbf{Y})_{av}$, where

$\mathbf{X} = \mathbf{K}\mathbf{X}_{fb}$ and $\mathbf{Y} = \mathbf{K}\mathbf{Y}_{fb}$, so that:

$$P = ((\mathbf{K}\mathbf{X}_{fb})^T(\mathbf{K}\mathbf{Y}_{fb}))_{av} = (\mathbf{X}_{fb}^T(\mathbf{K}^{T-}\mathbf{K}\mathbf{Y}_{fb}))_{av} \quad \text{(from (5))}$$

and substituting for $\mathbf{K}^{T-}$ from (4), therefore:

$$P = (\mathbf{X}_{fb}^T(\mathbf{K}^{-1}\mathbf{K}\mathbf{Y}_{fb}))_{av} = (\mathbf{X}_{fb}^T\mathbf{Y}_{fb})_{av} \qquad (6)$$

Oscillatory Transformation

The oscillatory power for real variables, according to the definition of the complex variables, is:

$$\mathbf{X}^T\mathbf{Y} = (\mathbf{K}\mathbf{X}_{fb})^T\mathbf{K}\mathbf{Y}_{fb} = \mathbf{X}_{fb}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{Y}_{fb} \qquad (7)$$

Example

Let $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be two real vectors.

Find out the oscillatory and time-average power transformation, then deduce the instantaneous power transformation.

The untransformed power is $\mathbf{X}^T\mathbf{Y} = x_1y_1 + x_2y_2$

but $\mathbf{X} = \mathbf{K}\mathbf{X}_{fb}$, $\mathbf{Y} = \mathbf{K}\mathbf{Y}_{fb}$, where

$$\mathbf{K} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -k & k \end{bmatrix}, \; \mathbf{X}_{fb} = \begin{bmatrix} x_f \\ x_b \end{bmatrix}, \; \mathbf{Y}_{fb} = \begin{bmatrix} y_f \\ y_b \end{bmatrix}$$

Therefore,

$x_1 = (x_f + x_b)/\sqrt{2} \quad x_2 = -k(x_f - x_b)/\sqrt{2}$

$y_1 = (y_f + y_b)/\sqrt{2} \quad y_2 = -k(y_f - y_b)/\sqrt{2}$

The oscillatory power = $\frac{1}{2}(x_f + x_b)(y_f + y_b)$

$$+k^2/2(x_f-x_b)(y_f-y_b) = x_f y_b + x_b y_f$$

$$\text{Time average power} = \mathbf{X}^{T^*}\mathbf{Y} = \tfrac{1}{2}(x_f+x_b)(y_f+y_b)$$

$$-k^2/2(x_f-x_b)(y_f-y_b) = x_f y_f + x_b y_b$$

$$\text{Instantaneous power} = (\mathbf{X}+\mathbf{X}^*)^T\mathbf{Y} = (x_f+x_b)(y_f+y_b)$$

$$= \text{time-average} + \text{oscillatory powers.}$$

Let the system be perturbed at frequency β and $\mathbf{Y}$ oscillates, so that: $y_1 = a\sin\beta t$, $y_2 = b\sin\beta t$. Let $\mathbf{X} = \mathbf{D}\mathbf{Y}$, as viewed with respect to time-varying reference frame of frequency ω, where:

$$D = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}. \text{ Find } \mathbf{X}, \mathbf{Y}_{fb} \text{ and } \mathbf{X}_{fb}.$$

$$\mathbf{X} = \begin{bmatrix} a\cos\omega t - b\sin\omega t \\ a\sin\omega t + b\cos\omega t \end{bmatrix} \sin\beta t$$

$$\mathbf{Y}_{fb} = \mathbf{K}^{-1}\mathbf{Y} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & k \\ 1 & -k \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \sin\beta t$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} a\sin\beta t + b\cos\beta t \\ a\sin\beta t - b\cos\beta t \end{bmatrix}$$

$$\mathbf{X}_{fb} = \mathbf{K}^{-1}\mathbf{X} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & k \\ 1 & -k \end{bmatrix} \begin{bmatrix} a\cos\omega t - b\sin\omega t \\ a\sin\omega t + b\cos\omega t \end{bmatrix} \sin\beta t$$

$$\begin{bmatrix} x_f \\ x_b \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a\sin(\beta+\omega)t + b\cos(\beta+\omega)t \\ 0 \end{bmatrix} \quad \text{for } \beta > \omega$$

$$\text{or} \quad = \sqrt{2} \begin{bmatrix} a\cos\omega t\sin\beta t - b\sin\omega t\sin\beta t \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} a\sin(\omega+\beta)t + b\cos(\omega+\beta)t - a\sin(\omega-\beta)t - b\cos(\omega-\beta)t \\ 0 \end{bmatrix} \quad \text{for } \beta < \omega$$

In case of $\beta < \omega$ we found $x_b = 0$, while x_f is the sum of both $x_f + x_b$ of the case $\beta > \omega$. This indicates that the backward wave in that frame exists when $\beta > \omega$, and becomes forward when $\beta < \omega$.

CONCLUSION

It is believed that, in the process of transformations between reference frames, use of the i-operator in conjunction with the complex variables is an encumbrance that hinders clear understanding and invites lack of rigour. A new 'k' operator is introduced, which is the 90° forward operator for real variables. The algebra of this operator is developed, and it is seen to work conveniently and clearly with variables of multiple frequencies. By its use, for example, it is seen for the first time that a 'backward' wave, as viewed in one spatial reference frame, becomes a 'forward' wave in another, as would be physically expected; but i-operator is faulty in this respect.

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