A FINITE STUDY OF PUMPING FROM GRAVITY WELL IN AN OTHERWISE UNDISTURBED AQUIFER

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ABSTRACT

A finite element numerical model has been designed in order to study the characteristics of steady state groundwater flow in the neighbourhood of deep gravity wells. Particular attention is paid to the flow pattern close to the well where the slope of the free surface is steep and the vertical velocity components generate a surface of seepage. The results are presented graphically in chart form suitable for practical use and are compared with results that can be obtained using non-numerical methods.

NOMENCLATURE

- A radial area of a domain
- $\{f^e\}$ element force vector
- $\{F\}$ global force vector
- h_s height of the seepage face
- h_w water depth in the well
- H_o original water depth in the aquifer
- k coefficient of permeability
- $[k^e]$ element stiffness matrix
- $[K]$ global stiffness matrix of a domain
- n number of nodes in element
- N_i shape function
- q_o rate of discharge per unit length
- Q well discharge
- r_w well radius
- R_o radius of influence
- s_A, s_B line boundary on which boundary conditions types (A) and (B) are imposed respectively
- s'_B part of the boundary on which boundary condition type (B) is imposed
- V volume
- Φ potential head
- Φ_p prescribed values of head
- $\{\Phi^e\}$ vector of unknown of element
- $\{\Phi\}$ global vector of unknowns to be determined

INTRODUCTION

The analysis of groundwater flow into gravity wells in unconfined aquifers is complicated because an important boundary condition, the position of the free surface, is initially unknown and has to be determined by an iterative process. A further complication arises from the occurrence of the seepage face , which always exists around the well casing and screen in unconfined aquifers, even if head losses due to entry of water into the well are ignored (see Figure (1)). The groundwater flow intersects the well at some distance above the water level in the well and emerges from the porous medium into the internal space, trickling down along the seepage face. The length of this face is unknown at the start of analysis and so is the shape of the free surface, particularly the point at which it joins the water table.

Dupuit [12] studied the regime of flow into a gravity well by neglecting the seepage face height at the well. Consequently, he derived the following equation:

$$Q = \pi k \, (H_o^2 - h_w^2)/\{\ln(R_o/r_w)\} \qquad (1)$$

or

$$Q = \pi k \, (H_o^2 - h^2)/\{\ln(R_o/r)\} \qquad (2)$$

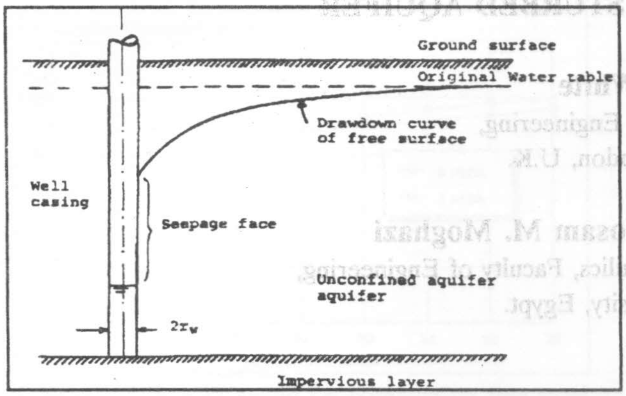

Figure 1. Formulation of the seepage at a pumped well in an unconfined aquifer.

where H_o is the original water depth in the aquifer, h_w is the water depth in the well, R_o is the radius of influence, r_w is the well radius and h is the water depth at radial distance r from the well. Equations (1) and (2) are used to obtain the well discharge and the profile of the free surface respectively. However, there are some objections to the Dupuit solution because; (1) it does not give the proper shape of the free surface in the area adjacent to the well, (2) it fails to give a flux value when h approaches zero, making the cross sectional area of flow at the well circumference zero, (3) it is based on the assumption that the ground water flow takes place in the horizontal planes, whereas in the immediate vicinity of the well a strong curvature of the flow lines is noticeable. Kashef [8] derived algebraic equations to obtain the position of the free surface based on the analysis of the neutral hydraulic forces acting on vertical prisms of the saturated soil. However according to Soliman [14] the capillary fringe, which was neglected by Kashef, should be taken into account.

Hansen [6] and Zee et al [17] have obtained different relationships to determine the seepage face height experimentally. However, their results are suitable for shallow aquifers rather than deep ones. Meanwhile some others, such as Hall [5] and Babbitt [2] were based on small ranges of the design variables.

Boreli [3], Kashef [9] and Rushton [13] applied the finite difference method to study the form of the free surface in the neighbourhood of the well. Taylor and Brown [15] used the finite element method (FEM) to investigate two dimensional study state flow through dams. They have presented an iterative approach to obtain the position the free surface. Although [7], [16], [4] and [10] approached the free surface flow problems successfully using the FEM they did not obtain expressions or relationships associated with the determination of the seepage face height, free surface profile or discharge of a gravity well.

The main objective of the present study is to attempt to confirm the relationship between the well hydraulic parameters, in the steady state, by the use of the finite element method. These are the position of the free surface, the seepage face height the quantity of discharge. In addition, the determination of the velocity distribution pattern near the well will also be investigated.

ASSUMPTIONS

In order to simplify the numerical study, certain assumptions are introduced. These are; (1) the well is pumped at a constant rate and the water bearing medium is assumed to be homogeneous and isotropic; (2) the flow is laminar and obeys the Darcy law; (3) the free surface is maintained at a constant level at a finite distance away from the well at radial distance equal to radius of influence; (4) the soil is fully saturated and the compressibility of both water and soil are neglected; (5) the effect of capillary flow in the zone above the free surface is neglected; (6) the well penetration is full and the head losses through both the well screen and the casing are neglected.

THEORETICAL CONSIDERATIONS

The general governing partial differential equation for steady state flow in an anisotropic and homogeneous porus continuum can be described as

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial \Phi}{\partial z}\right) = 0 \qquad (3)$$

where Φ is the potential head and k_x , k_y and k_y are the permeability coefficients in the cartesian x, y and z axis respectively. Since the domain and the flow are symmetrical about the well center line axis the cartesian coordinates (x, y and z) in equation (3) are transformed to cylindrical coordinates (r, z). Then the appropriate flow equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(k_r r \frac{\partial \Phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial \Phi}{\partial z}\right) = 0 \qquad (4)$$

where r and z are the radial and axial coordinates respectively, k_r and k_z are the permeability coefficients in r and z directions respectively.

For isotropic soil $k_r = k_z = k$, thus equation (4) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr \frac{\partial \Phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial \Phi}{\partial z}\right) = 0 \qquad (5)$$

The following two boundary conditions for equation (5) are generally encountered in groundwater flow:

(A) Specified head boundary condition, where the head to be specified at a nodal point on the boundary S_A

$$\Phi = \Phi_p \tag{6}$$

where Φ is the potential head and Φ_p is the prescribed head.

(B) Specified flux boundary, where a specified amount of flux q_o flows into the body, per unit length of the boundary S_B

$$k \frac{\partial \Phi}{\partial n} + q_o = 0 \tag{7}$$

where $\partial/\partial n$ is the outward pointing normal derivative to the boundary and q_o is specified discharge into the flow domain per unit length of the boundary.

By applying the Galerkin residual approach [11] to equations (5) and (7).

$$\int_V \left\{ \frac{1}{r}\frac{\partial}{\partial r}\left(kr \frac{\partial \Phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial \Phi}{\partial z}\right) \right\} N_i \, dV - \int_{S'_B} \left(q_o + k \frac{\partial \Phi}{\partial n}\right) N_i \, dS' = 0 \tag{8}$$

in which N_i is the shape function. It is appropriate to take

$$dV = 2\pi r dr, \; dz = 2\pi r dA \; \text{ and } \; dS' = 2\pi r dS \tag{9}$$

where V is the axisymmetric volume and S'_B denotes the part of the boundary on which boundary condition type (B) applies.

Substituting from equation (9) in equation (8) and applying the Green-Gauss theorem; then

$$\int_{S_A} \frac{\partial \Phi}{\partial x} N_i k r dS - \int_A \left(\frac{\partial N_i}{\partial r}\frac{\partial \Phi}{\partial r} + \frac{\partial N_i}{\partial z}\frac{\partial \Phi}{\partial z} \right) k r dA - \int_{S_B} q_o r N_i dS = 0 \tag{10}$$

The unknown head, Φ, in equation (10) may be approximated as [18]

$$\Phi = \sum_{i=1}^{n} N_i \Phi_i \tag{11}$$

In which Φ_i is the nodal values of Φ, where all or some of the parameters are unknown. Substituting for Φ, from equation (11) in equation (10), (see [11]), we obtain

$$\int_A kr \sum_{j=1}^{n} \left(\frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r} + \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z} \right) dA \, \Phi_j = X_i - \int_{S_B} q_o \, r \, N_i \, dS \tag{12}$$

where A = area of the domain

$$X_i = \int_{S_A} N_i \, kr \sum_{j=1}^{n} \frac{\partial N_j}{\partial r} \Phi_j \, dS \tag{13}$$

n = number of nodes in the element.

Equation (12) can be expressed in matrix form as

$$[k^e] \{\Phi^e\} = \{f^e\} \tag{14}$$

in which the element stiffness matrix $[k^e]$ and force vector $\{f^e\}$ are defined by

$$k_{ij} = k_{ji} = \int_A \left(\frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r} + \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z} \right) kr \, dA \tag{15}$$

and

$$f_i = X_i - \int_{S_B} q_o r N_i \, dS \tag{16}$$

The term X_i in equations (12) and (16) represents the flow at each nodal point associated with the prescribed head values along the boundary S_A. The previous procedure to obtain equation (14) is repeated for all the elements in the domain. Then all the equations are combined into a set of simultaneous equations

$$[K]\{\Phi\} = \{F\} \tag{17}$$

where [K] is the global stiffness matrix, $\{\Phi\}$ is the global vector of unknown head to be determined and {F} is the global nodal force vector. From equation (17), the final solution can be obtained after applying the boundary conditions.

BOUNDARY CONDITIONS

Referring to the geometry of the model in Figure (2), the boundary conditions associated with the flow towards a gravity well are as follows:

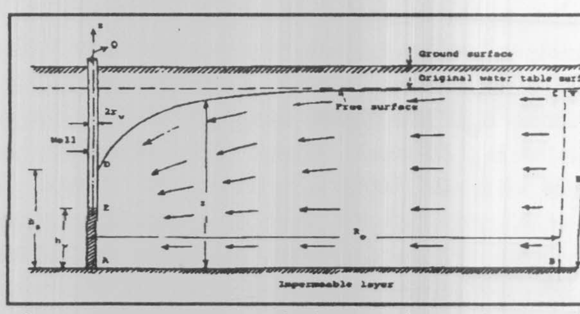

Figure 2. Geometry of the gravity well - aquifer configuration.

Impervious boundary:

Along the surface AB, $\frac{\partial \Phi}{\partial n} = 0$ (18)

Water boundaries:

These constitute the faces BC and AE. The potential head, Φ, along them equal to the elevation of the water face above the datum

$$\Phi = H_o \text{ (along BC)}, \quad \Phi = h_w \text{ (along AE)} \tag{19}$$

Phreatic surface:

Along CD the total head equal the elevation head

$$\Phi = z \text{ (along CD)} \tag{20}$$

In addition, the flow across this boundary is zero, i.e.,

$$\frac{\partial \Phi}{\partial n} = 0 \text{ (along CD)} \tag{21}$$

Seepage face:

At which the pressure is atmospheric

$$\Phi = z \text{ (along CD)} \tag{22}$$

PROCEDURE

In order to carry out the previous finite element computation steps, a finite element program, LUSAS [1], at University of London Computer Center has been utilised. Since the exact position of the free surface is initially unknown, the Taylor and Brown [15] iterative process has been followed herein.

MODEL DIMENSIONS

Values of the original water depth in the aquifer, H_o, and the well radius, r_w, were chosen equal to be 50.0 and 0.20m respectively. A wide range of the radius of influence, R_o, (100-1000m) has been chosen. For each value of R_o, different values of the drawdown ratio (h_w/H_o=0.8, 0.6, 0.4, 0.2, 0.0) were studied. The permeability coefficient of the soil, k, was chosen equal to 0.001m/sec. Figure (3) shows the finite element mesh using axisymmetric triangular elements. For each value of R_o and h_w/H_o, the mesh was modified to meet the new dimensions of the model by changing the element size.

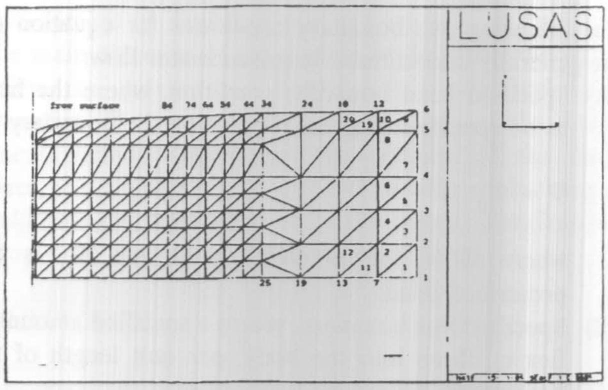

Figure 3. Finite element mesh (Axisymmetric elements).

RESULTS

Free surface

Figure (4) shows the profile of the free surface obtained for R_o/r_w=500 and h_w/H_o=0.0. Curves plotted from the equations proposed by Boreli [11], Dupuit [12], Babbitt [2] and Hall [5] are also shown in Figure (4). Generally, it can be seen from this figure that the FEM results agree well with those obtained by Boreli and Babbitt in particular, and the discrepancy between them does not exceed 0.5% at a radial distance (r=0.1 H_o). Comparison between the FEM and the Dupuit results shows that the Dupuit equation gives reasonably accurate results of the free surface profile at distance from the well greater than one half the original saturated depth of permeable stratum. However, the free surface is situated at a higher level than that obtained from the Dupuit's formula at a distance less than 0.5 H_o. It can also be seen from Figure (4) that there are large deviations between the profile of the free surface obtained by Hall and other profiles obtained by the FEM, Boreli and Babbitt. This is mainly due to the unsaturated flow and capillary effects were taken into account in the derivation of the Hall formula.

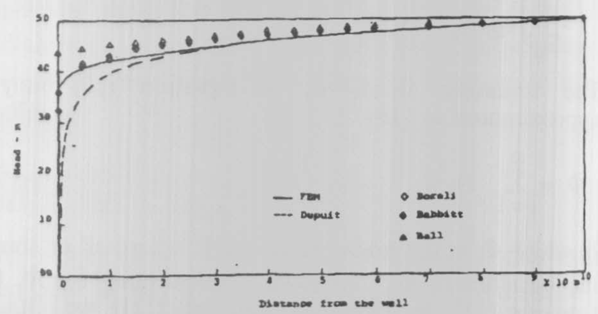

Figure 4. Comparison between the FEM and other methods to determine the profile of the free surface (R_o=100m, r_w=.2m, H_o= 50m, h_w=0.0m).

Seepage face

Figure (5) shows the variation of $(h_s-h_w)/H_o$ and R_o/H_o for various values of h_w/H_o, where h_s represents the seepage face height. It can be seen that the seepage face (h_s-h_w) is directly proportional to the drawdown ratio h_w/H_o and reaches a maximum as $h_w/H_o=0.0$. Meanwhile, it may be pointed out that the height of the seepage face, h_s, is bigger than half the original water depth. This means that it is impossible to lower the water level in an unconfined aquifer by more than 0.5 H_o .A comparison between the FEM, Babbitt [2] and Boreli [3] for different values of h_w/H_o is shown in Figure (6). It can be seen that a good agreement is noticed between the FEM and Boreli results compared to Babbitt's ones. This may be attributed to the fact that in the electric analogy study by Babbitt, the wedge which represented the well was very thin and any small cracks or lack of homogeneity as well as improper electrode contact will affect the accuracy of results in the region near the well.

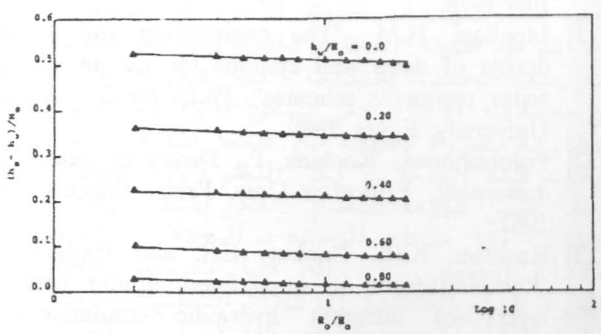

Figure 5. Relationship between the height of the seepage face, h_s, and h_w, H_o, R_o (H_o=50m, r_w=.2m).

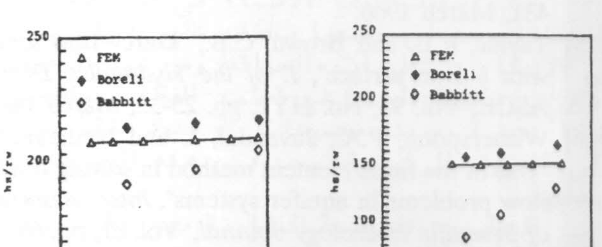

Figure 6. Comparison between the FEM and others for determining the height of the seepage face.

Discharge

Figure (7) shows the variation of well discharge Q and other parameters. A comparison between the FEM and the Dupuit formula is also shown in Figure (7). It can be seen that the results of both methods converge at small drawdown and diverge with the increase of the drawdown until maximum divergence is noticed at a maximum drawdown ($h_w/H_o=0.0$). This is attributed to the Dupuit assumption, where horizontal flow is assumed near the well and the vertical velocity near the well is ignored. However, the error in applying the Dupuit formula does not exceed 3% and 6% at $h_w/H_o=0.8$ and 0.0 respectively.

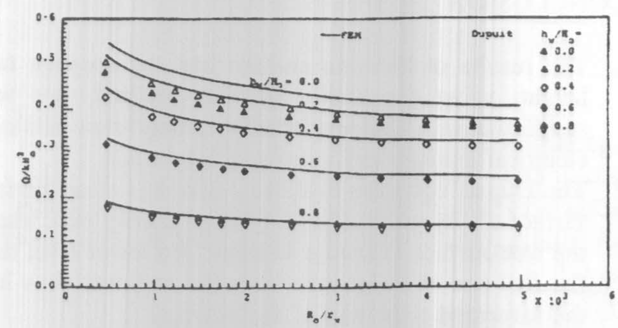

Figure 7. Comparison between the FEM and the Dupuit formula to determine well discharge.

It is worthwhile noting that knowledge of any two of the variables Q/kH_o^2 , R_o/r_w or h_w/H_o makes possible the prediction of the other variable from Figure (7).

Vertical Velocity Distribution

Figure (8) shows the variation of the vertical hydraulic gradient along the free surface $\partial\Phi/\partial z$, as obtained from the LUSAS output, and the radial distance ratio r/H_o from the well axis. The vertical velocity can also be obtained from this figure by multiplying the values of $\partial\Phi/\partial z$ by the permeability coefficient k. It is clear that the vertical hydraulic gradient increases in the radial direction towards the well and reaches a maximum value of unity at the well face. This means that the free surface must approach the well tangentially with a vertical velocity of k and a zero horizontal velocity. It can be concluded from this figure that the vertical velocity starts to decrease rapidly with increasing radial distance from the well axis and nearly vanishes at a radial distance between 0.5-0.6 H_o. Consequently, the Dupuit assumptions are not accurate enough within 0.5 H_o from the well axis, but can be used with sufficient accuracy outside this range.

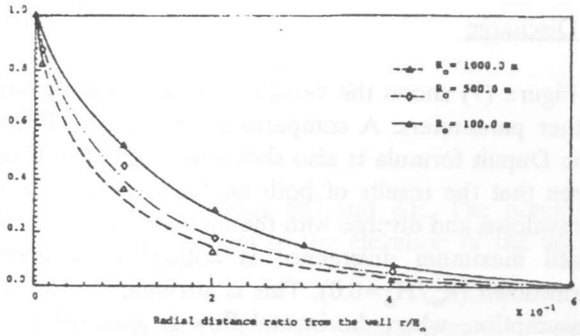

Figure 8. Variation of the vertical component of the hydraulic gradient near the well ($h_w/H_o=0.0$).

CONCLUSION

1. The results of the free surface and the seepage face height, using the finite element method, are well supported with the results of the membrane analogy, electrical analogy and relaxation approach.
2. The Dupuit equation cannot be used to define the free surface of the water adjacent to the gravity well, where the free surface is always situated at a lower level than the real one at a distance from the well less than half the saturated thickness of the aquifer.
3. In theory, it is impossible to lower the level in an unconfined aquifer by more than half the saturated thickness of the aquifer.
4. The vertical velocity components are of great significance near the well and decrease as the radial distance increases from the well, and nearly vanishes at a radial distance between 0.5-0.6 of the saturated thickness of the aquifer from the well axis. Thus, the Dupuit assumptions can be used accurately outside this range.
5. Although the finite element method is an approximate method, it has showed the accuracy and the flexibility to handle the free surface groundwater flow problem to a gravity well in an undisturbed aquifer.

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