Alex. Eng.J., Alex. Univ., Volume 28 No. 4, pp 373 - 391 (1989)

FRICTION FACTORS FOR PUDSATING LAMINAR FLOW

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Abstract

Prediction of friction factors for pulsating laminar flow in a rigid tube was obtained from the momentum equation and a sinusoidal driving pressure. The friction factor equation obtained is analogous to the Darcy-Weisbach equation with a new multiplier ($\pi/16\varphi$) which is dependent on the frequency of the Pulsation and the kinematic viscosity of the fluid. Friction factors calculated from well-defined experiments were compared with those predicted from the equation. It was found that the theoretical friction factors predicted the experimental values to-within 5 % for Womersley number up to 3 and to-within 8 % for Womersley number of 3-5.

Notation

C_1, C_2	constants
d	tube diameter
f, f_p	friction factors for steady and pulsating flow respectively
f(S), f(τ)	functions of time defined by Eqs. (9) and (12) respectively
g(S), g(τ)	functions of time defined by Eqs. (10) and (13) respectively
J_o, J_1	Bessel functions of the first kind of orders zero and one respectively
K	friction parameter defined by Eq. (4)
n	frequency (Hz)
p	pressure (N/m^2)
Δ p	pressure drop (N/m^2)
$(\Delta p)_{av}$	average pressure drop for pulsating flow (N/m^2)
q_n	roots of zero-order Bessel function
R	tube radius (cm)
Re	Reynolds number $2Ru/\nu$
r	radial coordinate
S	transformed time
t	time (s)
u	fluid velocity (cm/s)
u_{av}	average fluid velocity
w	transformed velocity
Y_o	Bessel function of the second kind of order zero
z	axial coordinate

Greek Letters

α Womersley number = $R \sqrt{\frac{\omega}{\nu}}$

ρ fluid density (gm/cm^3)

μ fluid viscosity ($gm/cm.s$)

ν kinematic viscosity (cm^2/s)

φ sum of infinite series defined by Eq. (18)

ω angular velocity = $2\pi n$

τ scaled time = $t - 3\pi/2$

1. Introduction

Pulsating flow of fluids occurs widely, both in nature and industry. Examples are circulation of blood flow in cardiovascular system, pumping of liquids and slurries and heat and mass transfer operations in extraction columns and heat exchangers.

The fluid dynamics of pulsating flow in a circular tube has been studied theoretically by Kusama [1], Lighthill [2], and Womersley [3] who derived the equation for average flow rate from the momentum equation and a sinusoidal pressure gradient. Interested in blood flow in arteries, Taylor [4] and McDonald [5] studied the propagation of a pulse wave in arteries and established the relation between pressure and flow in arteries using Womersley's theory [3].

For purpose of designing a pulsatile flow system, it is important to calculate the power requirements through knowledge of the friction at the tube wall. In laminar flow, Womersley [3], Alabastro and Hellums [6], Hershey and Song [7] have shown that the friction factors in pulsating flow are greater than the corresponding values for steady flow and depends strongly on the frequency of pulsation. In the case of turbulent flow, Streeter and Wylie [8] have found that a quasi-steady state model is satisfactory at frequencies up to 10 Hz. At higher frequencies, Brown et al [9] have observed that the flow pattern and turbulence did not have time to adjust to the rapid fluctuation. Recently, Round [10] and El-Masry and El-Shobaky [11] have shown , for slurry turbulent flow with superimposed pulsation, that the friction factors and energy dissipation were less than those for steady flow within a specified range of frequencies and amplitudes.

The present study is concerned with the development of a theoretical equation to predict the friction factors and hence the pressure drop for laminar pulsating flow with a sinusoidal pulse wave similar to the simple steady friction factor equation. An experimental test of the equation was undertaken to verify its applicability and limitations.

2. Theoretical Analysis

For laminar incompressible flow of Newtonian fluid in a rigid tube, the momentum equation for the z-component is given by

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right) \qquad (1)$$

If the upstream driving pressure is pulsating in the form of a sinusoidal wave P_o (1+sin ω t), then the pressure gradient dp/dz along the axis of the tube may be expressed by [3] a modified friction equation:

$$-\frac{dp}{dz} = \left(f_p \cdot \frac{\rho u^2_{av}}{2} \cdot \frac{1}{2R}\right)(1 + \sin\ t) \qquad (2)$$

where f_p is the friction factor for pulsating flow. Eq. (2) can be seen to reduce to the Darcy Weisbach equation when the frequancy is zero and f_p becomes f, the steady Darcy fiction factor.

If a new time scale is defined, $= t - \frac{3\pi}{2}$ and Eq.(2) is subistituted into Eq. (1), the results is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{\nu}\frac{\partial u}{\partial \tau} = -K(1 - \cos \omega\tau) \qquad (3)$$

where $K = f_p \frac{\rho_\rho u^2_{av}}{4\nu R}$ (4)

When Laplace transform is applied to Eq. (3) with respect to , Eq. (3) becomes

$$\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} - \frac{S}{\nu} w = -K\left(\frac{1}{S} - \frac{S}{\omega^2 + S^2}\right) \qquad (5)$$

where w is the transformed velocity. Eq. (5) is inhomogeneous zero-order Bessel differential equation with the solution in the form:

$$w(r,S) = C_1 J_o\left\{i\sqrt{\frac{S}{\nu}}\, r\right\} + C_2 Y_o\left\{i\sqrt{\frac{S}{\nu}}\, r + \frac{\nu K}{S}\left(\frac{1}{S} - \frac{S}{\omega^2 + S^2}\right)\right\} \qquad (6)$$

and the boundary conditions:

$$u(0,\tau) = \text{finite} \rightarrow w(0,S) = \text{finite}$$

$$u(R,\tau) = 0 \rightarrow w(R,S) = 0$$

Constants C_1 and C_2 can then be evaluated and Eq. (6) becomes

$$w(r,S) = \frac{\nu K}{S}\left(\frac{1}{S} - \frac{S}{\omega^2+S^2}\right)\left\{1 - \frac{J_o\left(i\sqrt{\frac{S}{\nu}}\,r\right)}{J_o\left(i\sqrt{\frac{S}{\nu}}\,R\right)}\right\} \tag{7}$$

The inverse Laplace Transform of Eq. (7) yields

$$u(r,\tau) = \frac{1}{2\pi i}\int_{C-i_\infty}^{C+i_\infty} w(r,S).\exp\,(S.\tau)\,dS \tag{8}$$

If we define f(S) and g(S) as:

$$f(S) = \nu K\left(\frac{1}{S} - \frac{S}{\omega^2+S^2}\right) \tag{9}$$

$$\text{and } g(S) = \frac{J_o\left(i\sqrt{\frac{S}{\nu}}\,R\right) - J_o\left(i\sqrt{\frac{S}{\nu}}\,r\right)}{S\,J_o\left(i\sqrt{\frac{S}{\nu}}\,R\right)}\exp\,(S\;\tau) \tag{10}$$

Then Eq. (8) can be solved for the inverse Laplace transform by the convolution integral:

$$u(r,\tau) = \int_o^\tau f(\tau')\,g\,(\tau - \tau')\,d\tau \tag{11}$$

From standard mathematical tables [12], Eq. (9) can be transformed to:

$$f(\tau) = K(1 - \cos \omega\tau) \tag{12}$$

and from the residue theorem the poles of g (S) in Eq. (10) could be obtained to give:

$$g(\tau)=2\sum_{n=1}^{\infty} \frac{J_o\left(q_n \frac{r}{R}\right)}{q_n J_1 q_n} \exp\left(-\nu q_{nR}^2 \tau/R^2\right) \tag{13}$$

where q_n = roots of zero order Bessel function = $i\sqrt{\frac{S}{\nu}}\cdot R$
and $J_o(q_n) = 0$

From Eqs. (11), (12) and (13)

$$u(r,\tau)=2\nu K\sum_{n=1}^{\infty} \frac{J_o\left(q_n \frac{r}{R}\right)}{q_n J_1(q_n)} \left\{ \frac{\nu R^2 q_n^2 (1-\cos\omega\tau) - R^4 \omega \sin\omega\tau}{\omega^2 R^2 + \omega^4 q_n^4} \right\} \exp\left(-\nu q_n^2 \tau/R^2\right) \tag{14}$$

Eq. (14) can be shown to reduce to the equivalent steady flow equation when the frequency goes to zero

Applying the mean value theorem to Eq. (14), the average velocity u_{av} can be obtained as:

$$u_{av} = \frac{\int_o^{2/\omega} \int_o^R ur\,dr\,d\tau}{(2\pi/\omega)\cdot \pi R^2} \tag{15}$$

or

$$u_{av} = \frac{2KR^2}{\pi} \sum_{n=1}^{\infty} \left[\frac{2\pi}{q_n^4} - \left\{ \frac{\alpha^6}{q_n^6(\alpha^4+q_n^4)} \right\} \left\{ 1 - \exp\frac{-2\pi\omega}{\alpha^2} q_n \right\} \right] \tag{16}$$

where α is the Womersley number [3] given by $\alpha = R\sqrt{\frac{\omega}{\nu}}$

Eq. (16) can also be shown to reduce to the Hagen-Poiseuille equation when the frequancy is zero. Subistitution of Eq. (4) into Eq. (16) and rearrangement. The friction factor in pulsatile flow (f_p) is shown to be:

$$f_p = \frac{2\pi\nu}{Ru_{av}} \cdot \frac{1}{\sum_{n=1}^{\infty} \left[\frac{2\pi}{q_n^4} \quad \frac{\alpha^6}{q_n^6(\alpha^4+q_n^4)} \right]} \tag{17}$$

If we define . as

$$\varphi = \sum_{n=1}^{\infty} \frac{2\pi}{q_n^4} \cdot \frac{\alpha^6}{q_n^6(\alpha^4+q_n^4)} \tag{18}$$

Then Eq. (17) becomes:

$$f_p = \frac{4\pi}{Re} \cdot \frac{1}{\varphi} = \left(\frac{\pi}{16\varphi}\right)\left(\frac{64}{Re}\right) \qquad (19)$$

Thus the friction factor for sinusoidal pulsatile flow f_p can be calculated by means of equation (19) through the average Reynolds number (Re) and the factor (φ) which is a function of the Womersley number. In Fig. (1) the factor (φ) and the multiplier value of ($\pi/16\varphi$) are plotted against . It is to be noted that for value of (α =0) i.e steady flow, the factor becomes $\frac{1}{16\varphi}$ and the multiplier equals unity. As (α) increases the factor (φ) decreases and the multiplier increases to approach ∞ at a value of (φ =0) at Womersley number of $\alpha = 6.192$. Beyond α =6.192, negative values of(φ) are obtained due to the nature of the expansion in Eq. (18).

The theoretical values of f_p at different values of α can therefore be compared with values obtained from well-defined experiments through the friction equation:

$$f_p = \frac{(\Delta P)_{av}}{\rho u^2_{av}/2} \frac{2R}{L} \qquad (20)$$

where $(\Delta P)_{av}$ is the time-averaged experimental pressure drop over length L of the tube. Curves for theoretical pulsatile flow friction factors are plotted in Fig. (2) as a function of Reynolds number for different values of Womersley number.

3. Experimental

The experimental apparatus was designed to produce, within a horizontal tube, a fully-developed laminar flow with a sinusoidal

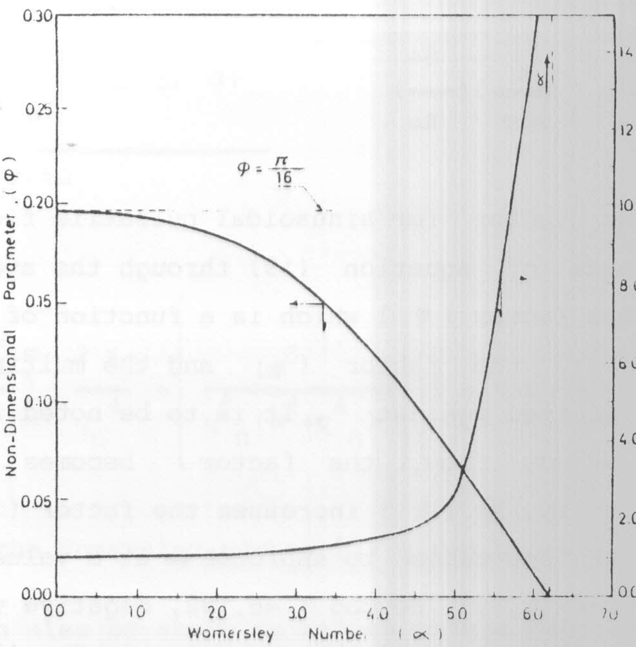

Fig. 1 Non-dimensional parameter (φ) and pulsating flow multiplier(π / 16 φ) Versus Womersley number (α).

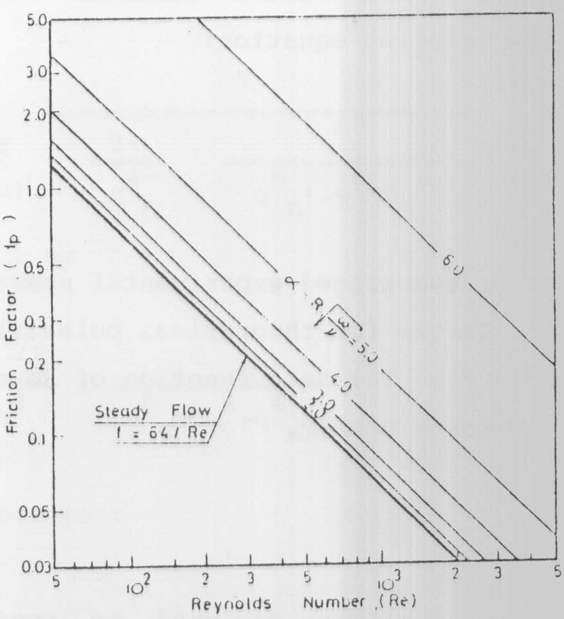

Fig. 2 Theoretical pulsating flow friction factor (f_p) versus Reynolds number (Re).

pressure pulsation superimposed on the steady component. A schematic diagram of the flow system is shown in Fig. (3). Two test sections of Pyrex tubes (A) with different diameters (0.635 abd 0.953 cm) were used. The tubes were 140 cm long with two holes drilled 75 cm apart for pressure taps (B). The tube length preceded the first tap was long enough to ensure a steady parabolic velocity profile in the test section over the experimental range of Reynolds number. Fluids from the test section and over flow from the constant head tank (C) where discharged into a reservoir (D) and recycled through a small centrifugal pump (E) to the constant head tank. Pulsation was generated by a pulsating piston (F) connected to the flow system through a T-piece. of Pyrex tubing and driven bq a small (1/6 hp.) variable speed motor (G) and a scotch-yoke mechanism. Both the

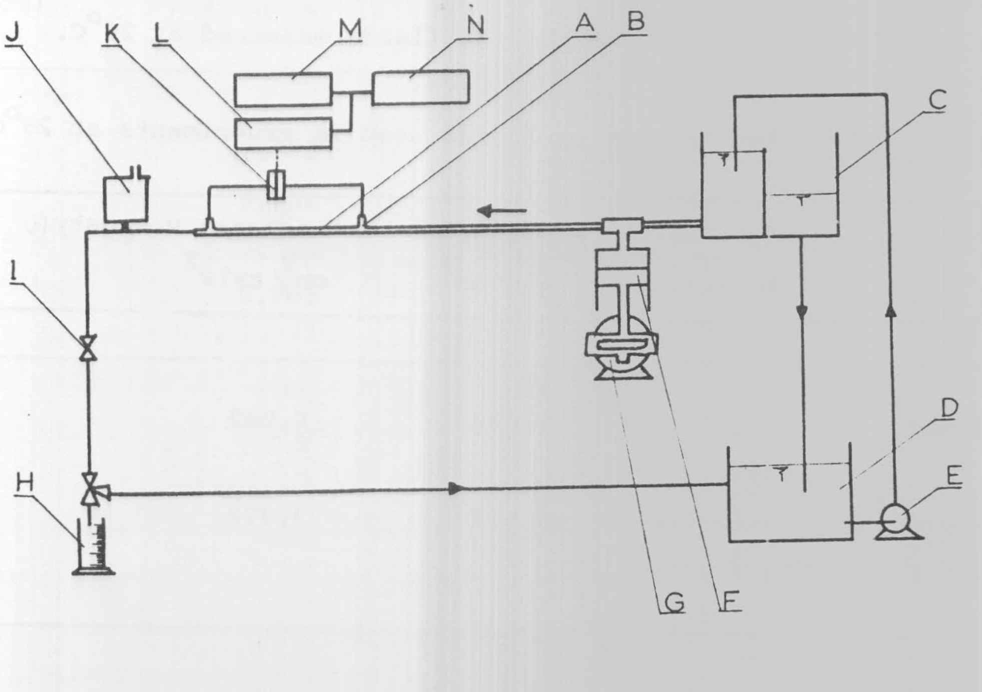

Fig. 3 Schematic diagram of the experimental apparatus.

frequency and amplitude of the pulse were controlled independently through adjustment of the motor speed and the scotch yoke stroke respectively. The steady component of flow in the system was measured by a graduated cylinder (H) and a stop-watch and controlled by a valve (I). A small inverted bottle was used in the system as a surge tank (J) to damp out any small fluctuations in the system.

The pressure drop across the test section was measured using a differential pressure transducer (K) (CELESCO model K P 15) connected to the pressure taps through Tygon tubing. The transducer output was fed to a reading unit (L), (CELESCO, model CD25C). The output signal of the reading unit was fed to an R.M.S Voltmeter (M), and/or a chart recorder (N), (HOLAND model BD40).

The circulating fluids used were (1) water for one phase of the experiments and (2) 42 per cent glycerine in water for the other to extend the range of the expenimental parameters. Table (1) gives detailes of the properties of the two fluids measured at 25°C.

Table (1) Properties of working fluids used in experiments at 25°C

Fluid	Composition by volume	Density(ρ) gm/cm^3	Kinematic viscosty(ν) $cm^2/s \times 10^2$
(1) Water	Distilled water	1.000	0.982
(2) Glycerine/ water	Glycerine 42% + water	1.112	3.772

Experinents were carried-out to cover a range of laminar steady Reynolds number (Re) of 200-1950. Using frequency range of 0.24-0.95 Hz, two different tube diameters and two different working fluids, it was possible to cover a wide range of Womersley numbers (α). Tables (2) and (3) give detailes of the experimental flow parameters.

In all experiments, steady flow component was first adjusted and measured, then the pulsation was superimposed with the predetermined values of frequancy and amplitude. measurements of $(\Delta p)_{av}$ from the R.M.S voltmeter was checked against the values obtained from the recorded pressure plot through digitization before using Eq. (20) to determine the friction factors.

Table (2) Flow velocities and Reynolds number for different experiments

Tube diameter d(cm)	Kinematic Viscosity $\nu\,(cm^2/s)x10^2$	Velocity u(cm/s)	Reynolds Number Re
0.635	3.772	11.8-47.5	200-800
0.635	0.982	12.4-30.2	800-1950
0.953	3.772	7.9-31.7	200-800
0.953	0.982	8.2-20.1	800-1950

Table (3) Frequencies and Womersley numbers for different experiments

Tube diameter d(cm)	Kinematic Viscosity $\nu(cm^2/s)x10^2$	Frequency n(Hz)	Womersley number α
0.635	3.772	0.24-0.95	2.00-400
0.635	0.982	0.25-0.56	4.00-6.00
0.953	3.772	0.24-0.95	3.00-6.00
0.953	0.982	$\geq$0.25	6.00 and up

4. Results and Discussion

All experiments carried-out in the present study were limited to the case where the amplitude of the flow rate was less than the steady component to avoid additional losses due to reversed flow. Typical pressure drop-time (Δ p-t) curves are given in Fig. (4) Measured values of (the steady flow pressure drop)(Δ p) are drawn on the figure.

The calculated values of the friction factor based on the $(\Delta p)_{av}$ by means of Eq. (20) are compared with the predicted theoretical values obtained from Eq. (19) in Figs. 5,6,7,8 and 9 for values of the Womersley number of 2,3,4,5 and 6 respectively. The friction factor predicted by Eq. (19) represented the experimental results quite well for values of Womersley number up to 5. The difference was less than 5 % for α = 2 and 3, and not more than 8 % for α = 4 and 5. On the other hand, the experimental friction values for α = 6 was lower than the theoretical values calculated by Eq. (19) by 20-25 %. These smaller than expected values may be attributed to the distortion of

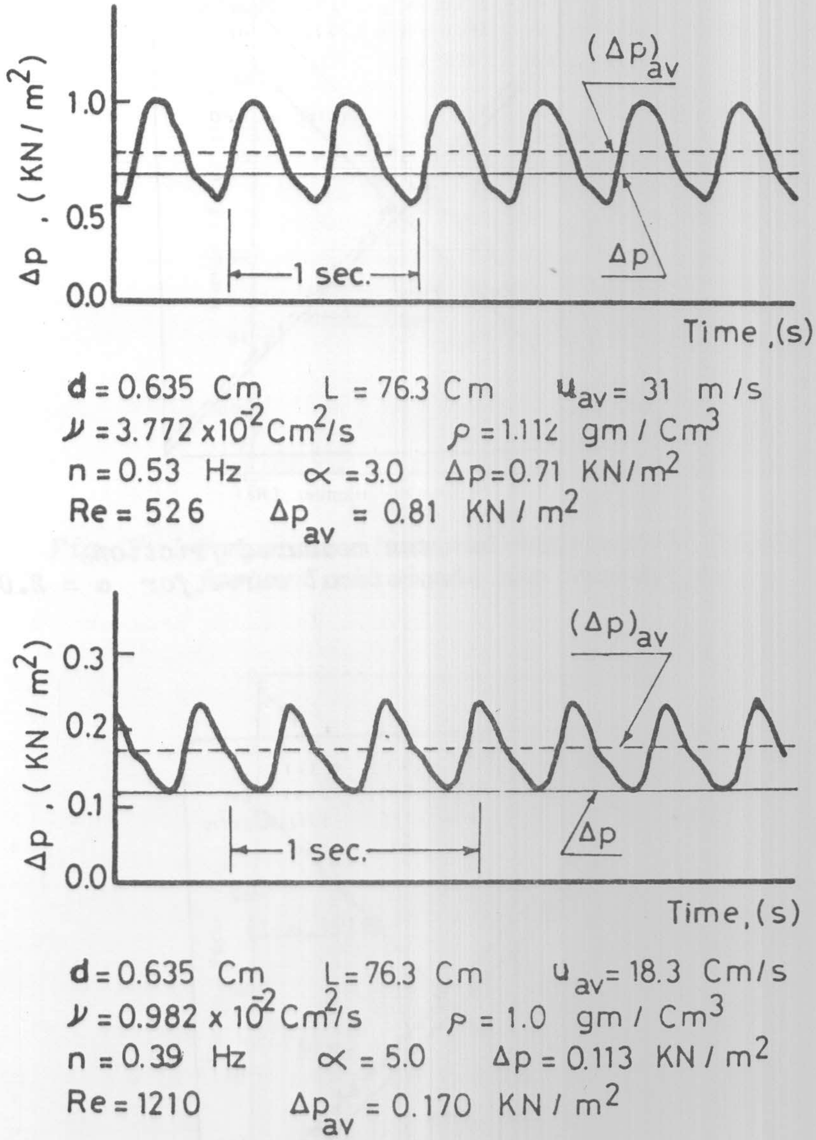

Fig. 4 A typical pressure drop-time (Δ p-t) curves for two different experiments.
Δ p = Steady flow pressure drop

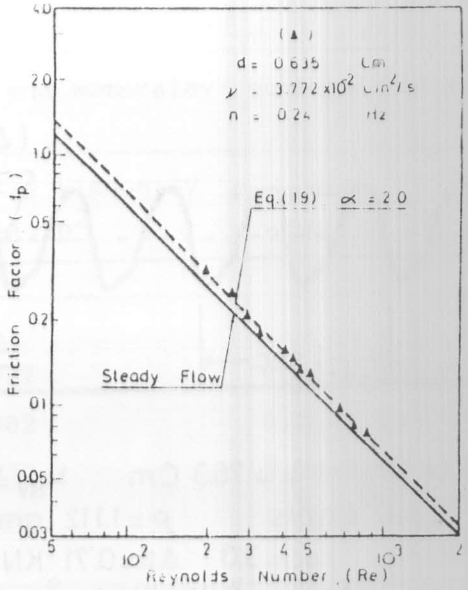

Fig. 5 Comparison between measured friction factors and theoretical curve for $\alpha = 2.0$

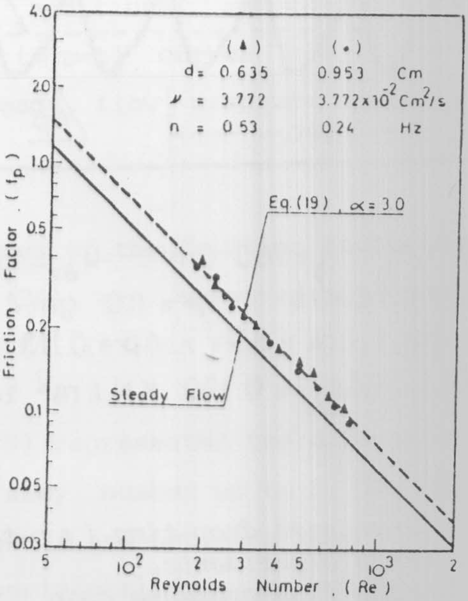

Fig. 6 Comparison between measured friction factors and theoretical curve for $\alpha = 3.0$

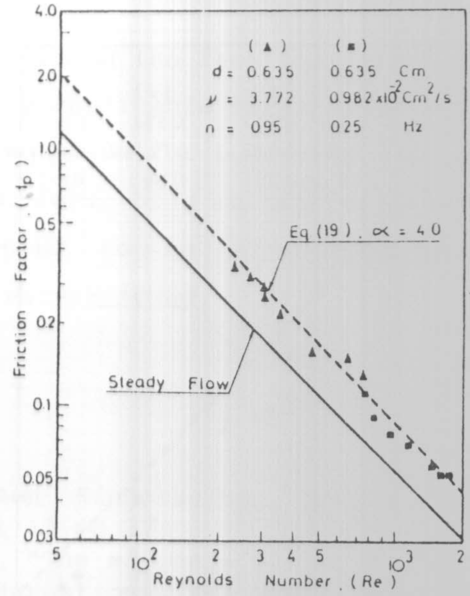

Fig. 7 Comparison between measured friction factors and theoretical curve for $\alpha = 4.0$

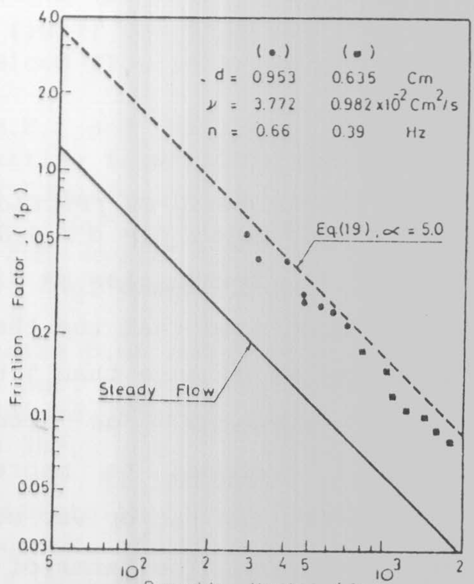

Fig. 8 Comparison between measured friction factors and theoretical curve for $\alpha = 5.0$

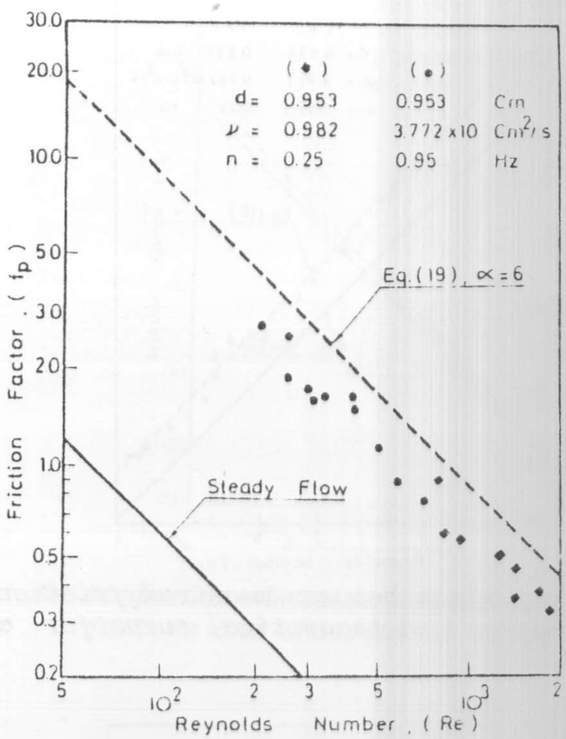

Fig. 9 Comparison between measured friction factors and theoretical curve for $\alpha = 6.0$

the boundary layer at high frequancies as discuesed by Shirotsuka [13]. However, it is to be noticed that the theoretical model should be used with caution for values of more than 5 because the multiplier ($\pi/16\ \varphi$) in Eq. (19) approachs as discussed in the analysis section. Shirotsuka [13] proposed to represent the fractional increments of pulsatile friction factor vs. steady flow values by a non-dimensional empirical equation. The terms of the equation included the frequency and amplitude of pulsation in addition to the tube diameter and the fluid viscosity. However, Shirotsuka's proposed equation involved extensive calculations and required steady flow values to determine the friction factors for the corresponding pulsating flow.

Conclusion

The present study confirmed the fact that the Darcy-Weisbach equation using f=64/Re is only applicable strictly to steady flow conditions. In order to use that equation for pulsating flow with (α) up to 5, a modification in the form of a multiplier given by Eq. (19) must be used which was proved to be more accurate and predicted the experimental results satisfactory.

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