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A GRAFICAL METHOD FOR THE ANALYSIS OF VARIABLE STRESSES

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ABSTRACT

Using vector analysis, a new graphical method is proved and modified to combine the sinusoidally varing normal and shear stresses which are, in general, out—of phase. The values and directions of the maximum normal and shear stresses at any instant and the maximum values of these stresses overall the load cycle obtained from one construction.

Some applications are demonstrated to show the advantage of this method and its capability to estimate the values of the applied loads and their phase angles to obtain certain function of the maximum stresses (constant for example).

NOMENCLATURE

σ	normal stress
τ	shear stress
A	amplitude of normal stress
B	amplitude of shear stress
w	circular frequency
t	time
ϕ	phase angle (between σ_x and σ_y)
γ	phase angle (between σ_x and τ_{xy})
$\hat{\sigma}$	unit vector in σ direction
$\hat{\tau}$	unit vector in τ direction
σ_1, σ_2	principal stresses
θ	direction of σ_1
τ_m	maximum shear stress

SUBSCRIPTS

x	co–ordinate in x–direction
y	co–ordinate in y–direction
xy	co–ordinate shear
—	free vectors
n	normal direction at τ_m plane
m. max	maximum value of τ_m
m. min	minimum value of τ_m

1- INTRODUCTION

The solution of the fatigue loading problem for monofrequency combined loads depends on whether such loads are in—or out—of phase. The inphase load problem have been carefully solved [1, 2, 3] form which it has been shown, for sinusoidal load that while the magnitude of the maximum normal and shear stresses varies sinusoidally, their directions remain constant.

In 1966 R.E. Little [4] proposed a graphical method to calculate the maximum shear stress for variable loads (Normal and shear) with different phase angle and frequency. This method is based on drawing the $\left(\tau_{xy} - \sigma_x\right)$ diagram on the x, y plane and to surround it with an ellipse (similar about x, y axis, its ratio 1:2). The point of tangency gives the maximum value of the maximum shear stress.

The outhor could not be able to find further work in the literature concerning this course of study of the out of phase loads.

The Mohr's circle as a graphical method gives instantaneous solution for the three variable stresses $\left(\sigma_x, \sigma_y, \tau_{xy}\right)$ out of phase problem. In this work the Mohr principle is generalized to cover all the load cycle in one graph.

PRESENT WORK

In the suggested method, the considered element is shown in Figure 1.

$$\sigma_x = A_x \cos (wt)$$

$$\sigma_y = A_y \cos(wt + \phi)$$

$$\tau_x = B \cos(wt + \gamma)$$

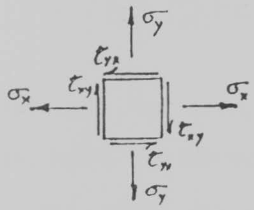

Fig. 1

where; A_x, A_y, B, ϕ, γ and w are constants.

Also, consider the following free vectors at the $\hat{\sigma}$, $\hat{\tau}$ plane Figure 2

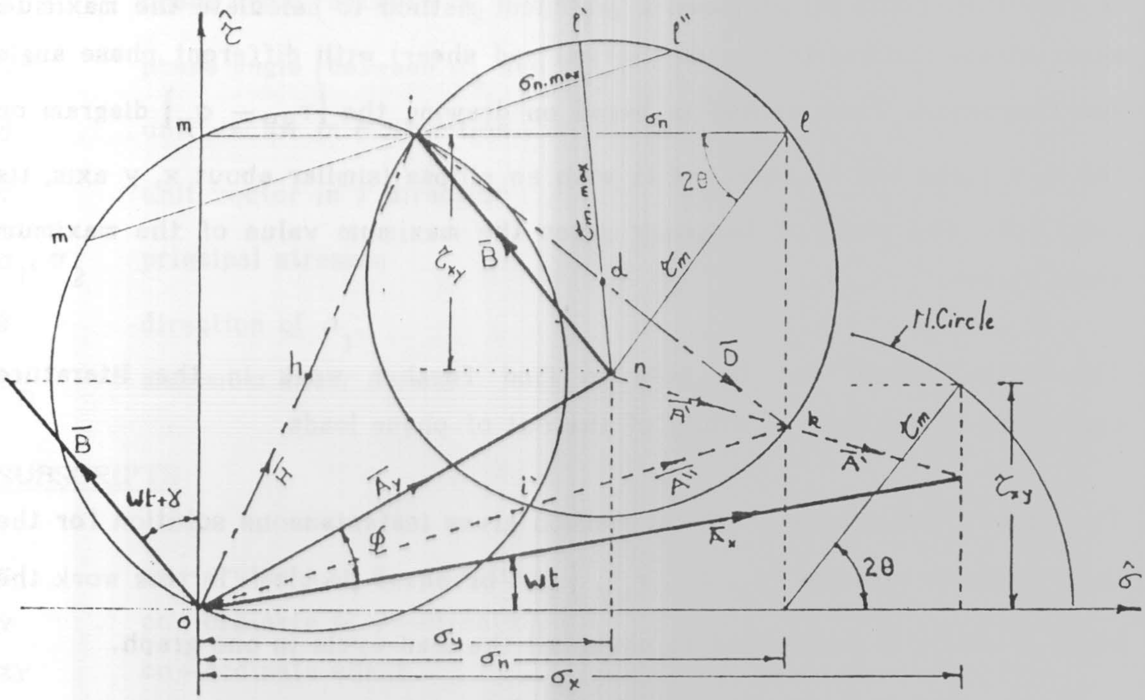

Fig. 2

$$\overline{A}_x = (A_x, wt)$$

$$\overline{A}_y = (A_y, wt + \phi)$$

$$\overline{B} = \left[B, wt + \gamma + \frac{\pi}{2} \right]$$

$$\bar{A}' = \frac{(\bar{A}_x - \bar{A}_y)}{2}$$

$$\bar{A}'' = \frac{(\bar{A}_x + \bar{A}_y)}{2}$$

$$\bar{D} = A' \quad \bar{B}$$

$$\bar{H} = \bar{A}_y + \bar{B}$$

$\hat{\sigma}$, $\hat{\tau}$ are unit vectors

Draw the circle "D" with diameter $|\bar{D}|$ and center at point "d"; also, the circle "H" qwith diameter $|\bar{H}|$ and center at point "h" The two circles intersect at two points "i, i'". Circle "H" intersects the "$\hat{\tau}$" axis at "m", angle $o\hat{m}i = \frac{\pi}{2}$, and the line "lim" is horizontal (parallel to $\hat{\sigma}$). Also, line "lk" is vertical (parallel to $\hat{\tau}$). At time "t" the maximum shear stress is determined as follows :

$$\tau_m = \left(\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2\right)^{\frac{1}{2}}$$

$$= \left\{\left(\frac{A_x \cos(wt) - A_y \cos(wt + \phi)}{2}\right)^2 + \left(B \cos(wt + \gamma)\right)^2\right\}^{\frac{1}{2}}$$

$$= \left\{\left(\frac{\bar{A}_x \cdot \hat{\sigma} - \bar{A}_y \cdot \hat{\sigma}}{2}\right)^2 + [\bar{B} \cdot \hat{\tau}]^2\right\}^{\frac{1}{2}}$$

$$= \left((\bar{A}' \cdot \hat{\sigma})^2 + [\bar{B} \cdot \hat{\tau}]^2\right)^{\frac{1}{2}} = \text{length of line "nl"}$$

The normal stress "σ_n" at the plane of "τ_m" is, then,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} = \frac{A_x \cos(wt) + A_y \cos(wt + \phi)}{2}$$

$$= \frac{\bar{A}_x \cdot \hat{\sigma} + \bar{A}_y \cdot \hat{\sigma}}{2} = \bar{A}'' \cdot \hat{\sigma}$$

$$= \text{length of line "ml"} \qquad (1)$$

The angle between "nl" and "$\hat{\sigma}$" ($n\hat{l}m$) is equal to "2θ" where "θ" gives the direction of principal stress "σ_1" at time "t" Figure 3.

Also, we have the following relations :

$$\left.\begin{aligned} \sigma_1 &= \sigma_n + \tau_m \\ \sigma_1 &= \sigma_n - \tau_m \end{aligned}\right\} \qquad (2)$$

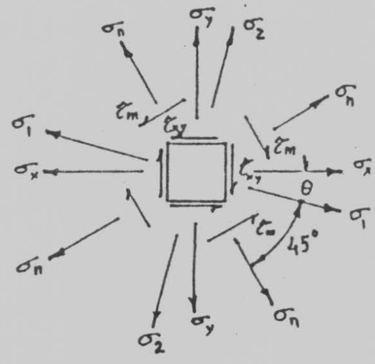

Fig. 3

It is abviously that all vectors $\left(\bar{A}_x, \bar{A}_y, \bar{B}, \bar{A}', \bar{A}'', \bar{H} \right.$ and $\left. \bar{D} \right)$ dose not change its angles relative to each other when changing time "t", i.e., line "mil" remains horizontal and the angles $\left(o\hat{m}i = i\hat{l}k = \frac{\pi}{2} \right)$. When rotating line "mil" clockwise, with "m" at circle "H" and "l" at circle "D", the following relations can be deduced.

$$\tau_m = n\,l$$

$$\sigma_n = m\,l$$

$$2\theta = m\hat{l}n$$

τ_m and σ_1 directions are shown in Figure 3 and "wt" is the angle between "ml" and "$\bar{A}_x$".

Now, it is clear that the maximum value of "τ_m" ($\tau_{m.max}$) is obtained when line "nl" passes through point "d" (ndl'). Also, the maximum value of σ_n ($\sigma_{n.max}$) is obtained, from equation (1), when line "mil" is parallel and equal to $\bar{A}''$ (m' i l''). The maximum value of "σ_1" is less than ($\sigma_{n.max} + \tau_{m.max}$)

$$\sigma_{1.max} \leq \sigma_{n.max} + \tau_{m.max}$$

To obtain the position of "$\sigma_{1.max}$" we have to find the corresponding point "l" which lies between the two points "l'" and "l''" by trail.

3- APPLICATIONS

The purpose of this graphical method is not only to obtain the values of the maximum stresses for normal case of loading but also to achieve a case of loading to obtain certain function of "τ_m, σ_n, σ_1, σ_2 and θ" overall the load cycle. This is demonstrated in the following examples :—

a— Constant maximum shear stress

It is clear from Figure 2 that the location of point "n" with "d" gives the variation in "τ_m". So if points "n " and "d" coincide, then, from Figure 4, :—

$$\tau_m = n\,l = \text{radius of circle "D"} = \text{constant} = |\bar{B}|.$$

Then , σ_1 will be maximum as σ_n is maximum

$$\sigma_{1.max} = \sigma_{n\ max} + B$$

where

$$\sigma_{n.\ max} = |\bar{A}''|$$

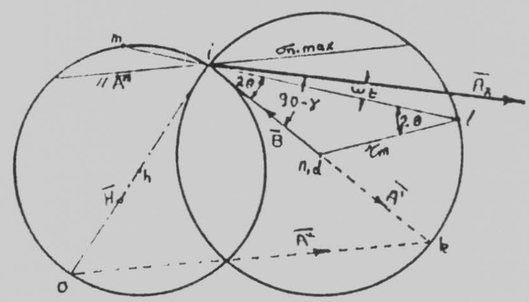

Fig. 4: ($\bar{B} = -\bar{A}'$, τ_m = constant)

It is clear that τ_m and σ_1 change their directions with time "t" according to the linear relation

$$2\,\theta = \frac{\pi}{2} - \gamma - wt$$

b— Constant principal stress

If the length "ml" is equal to zero the value of "σ_n" will be zero. This occurs when the two circles "D, H" coincide, and point "m" lies on point "l". In this case point "o" and "k" coincide, and the value of "σ_1" is equal to "τ_m".

If point "n" lies on the center of circles "D, H", Figure 5, then the value of "σ_1" is represented by the radius of circle "D" or "H". In this case points "n, h and d" coincide then, :-

$$\sigma_1 = \tau_m = |\bar{B}|$$

$$\sigma_n = 0$$

$$2\theta = -wt$$

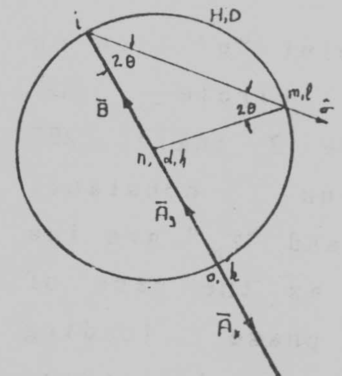

Fig. 5: ($\bar{A}_x = -\bar{A}_y = -\bar{B}$, $\sigma_1 = \tau_m$ = Constant, $\sigma_n = 0$, $2\theta = -wt$)

This mean that $\phi = 180°$, $\gamma = 90°$ and $A_x = B = A_y$, or in vector form $\bar{A}_x = -\bar{A}_y = -\bar{B}$

It is interesting to notice that if $\bar{A}_x = -\bar{A}_y = \bar{B}$ (vector $\bar{B}$ is inversed) points (h, d, o, i, m, l and k) coincide and the radius of circles "D, H" is zero, Figure 6

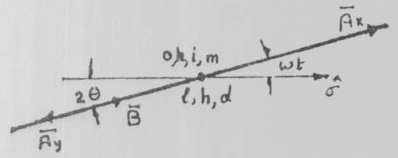

Fig. 6: ($\bar{A}_x = -\bar{A}_y = \bar{B}$, $\tau_m = \sigma_1 = \beta =$ Constant, $\sigma_n = 0$, $2\theta = wt$)

c– The directions of principal stresses is constant

If point "n" lies on the circle "D" Figure 7 angle "2θ" remains constant, "σ_1" and "τ_m" are the same as the case of in phase loading (constant directions and variable values)

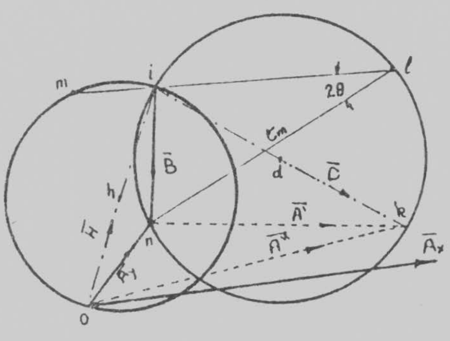

Fig. 7: (θ = constant)

4- Modified Method

In the previous method it is possible to obtain the values of "τ_m, σ_n" directly. The values of "σ_1, σ_2" should be calculated according to equation 2.

In this proposed method it is possible to obtain (σ_1, σ_2, τ_m and σ_n) directly from one figure.

Draw the vectors "$\bar{A}_x$, $\bar{A}_y$, $\bar{A}'$, $\bar{A}''$, $\bar{B}$ and $\bar{D}$" as in Figure 8. Locate a mid point p on the middle of $\bar{A}''$. The angle "$o\hat{q}k$"

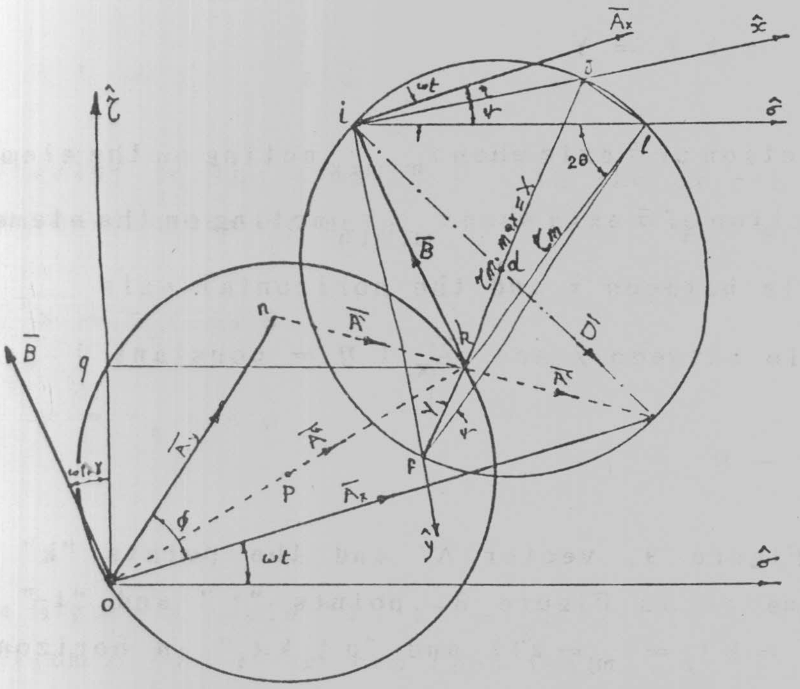

Fig. 8

is always equal to "$\pi/2$", circle "A''" give the path of point "q" and line "qk" is always horizontal. From equation (1)

$$\sigma_n = \bar{A}'' \cdot \hat{\sigma} = q\,k$$

Draw circle "D" with diameter $|\bar{D}| = |\bar{B} - \bar{A}'|$ as in Figure 8, then draw "il" horizontal, where "l" at circle "D", we know that

$$k\,l = \left[\left(\bar{A}' \cdot \hat{\sigma} \right)^2 + \left(\bar{B} \cdot \hat{\tau} \right)^2 \right]^{\frac{1}{2}} = \tau_m$$

$$\tau_{m.max} = kdj = X$$

$$\tau_{m.min} = k\,f = Y$$

if $\hat{x}$ is the direction of $\hat{\sigma}$ axis when $\tau_{m.max}$ acting on the element

$\hat{y}$ is the direction of $\hat{\sigma}$ axis when $\tau_{m.min}$ acting on the element

ψ is the angle between $\hat{x}$ and the horizontal axis

η is the angle between $\hat{x}$ and $\bar{A}_x$ (η = constant)

$$\psi = wt - \eta$$

Now consider Figure 9, vector $\bar{A}''$ and the points "k" and "q" be the same as in Figure 8, points "t_1" and "t_2" are such that ($k\,t_1 = k\,t_2 = \tau_m = kl$) and "$q\,t_2\,k\,t_1$" is horizontal line

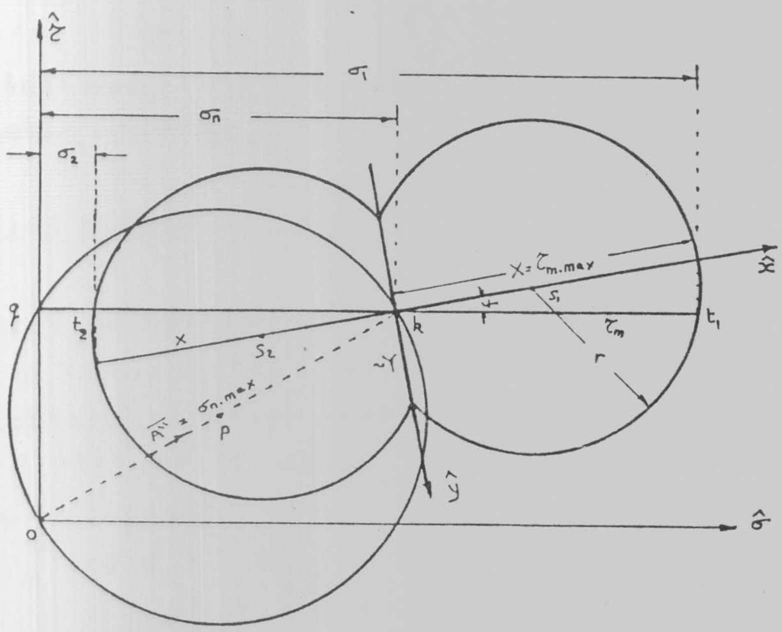

Fig. 9: ($wt = \eta + \psi$, η = constant, $\sigma_1 = q\,t_1$, $\sigma_2 = q\,t_2$
$\sigma_n = q\,k$, $\tau_m = k\,t_1$)

$$q\ t_1 = \sigma_1 \qquad \text{and} \qquad q\ t_2 = \sigma_2$$

Reffering to the axes $\hat{x}$, $\hat{y}$ with the origin "k", Figure 9, the coordinates of point " t_1 " (x , y) will be

$$\left.\begin{aligned} x &= \tau_m \cos\psi \\ y &= \tau_m \sin\psi \end{aligned}\right\} \tag{3}$$

from Figure 8 angle "j $\hat{l}$ f" = $\frac{\pi}{2}$, "j $\hat{f}$ l" = "j $\hat{i}$ l" = ψ and from Δ "j l f" we can find that :

$$\tau_m^2 = Y^2 \sin^2\psi + X^2 \cos^2\psi$$

from equation (3)

$$\tau_m^2 = \frac{Y^2\, y^2}{\tau_m^2} + \frac{X^2\, x^2}{\tau_m^2}$$

$$(x^2 + y^2)^2 = Y^2\, y^2 + X^2\, x^2 \tag{4}$$

Equation (4) gives a symmetrical curve with $\hat{x}$ and $\hat{y}$ axes. If Y = 0 it will be two circles with centers at $\pm\frac{X}{2}$ and diameter X, also if X = Y it will be one circle with diameter "2X" and its center at k, now if $1 > \frac{Y}{X} > 0$ it can be approximated to two circles with radius "r" and centers "S_1, S_2" where $\frac{X}{2} > k\,S_1 > k\,c$, and "c" is the center of a circle intersects the $\hat{x}$ axis at X and $\hat{y}$ axis at $\pm$ Y its radius $\rho = \frac{[X^2 + Y^2]}{2X}$ Figure 10.

We find that S_1 must be between these two points "u, c" with the ratio (X–Y) : X to give the best approximation

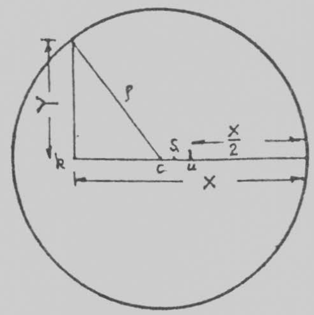

Fig. 10

$$\frac{c\,S_1}{S_1 u} = \frac{[X-Y]}{X}$$

Let $R = \frac{Y}{X}$, we can fined that :

$$\frac{k\,S_1}{X} = \frac{(1-R)\,(2+R)}{2\,(2-R)}$$

$$\frac{r}{X} = \frac{[2 - R - R^2]}{2\,(2-R)}$$

Figure 11 show the error "E" using this approximation for different values of $\frac{Y}{X}$. We can see that $E < 1\%$ in the most important area where $\tau_{m.max}$ and $\sigma_{1.max}$ positions.

Now for any time "t", draw from "k" a line making angle "wt" with $\bar{A}_x$, that line "$qt_2\ kt_1$" Figure 9 give the values of σ_1, σ_2, σ_n and τ_m

The value of $\sigma_{1.max}$ can be obtained directly from drawing by turning the line "qt_1" to get its maximum or by calculating its angle ψ_1, from the equation $\frac{\partial \sigma_1}{\partial \psi} = 0$ (Appendix 1)

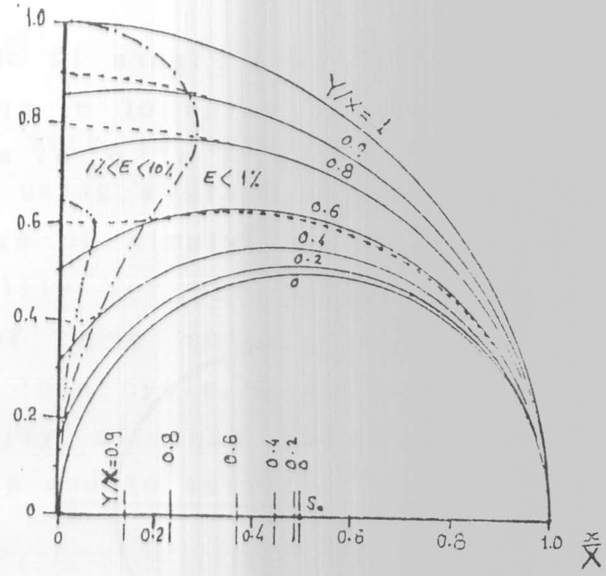

Fig. 11: $\frac{Y}{X}$ =	0	0.2	0.4	0.6	0.8	0.9	1
$\frac{r}{X}$ =	0.5	0.511	0.55	0.629	0.767	0.868	1

$$\psi_1 = \sin^{-1}\left[\frac{\left(\frac{r}{a}\right)^2\left(V\tan\psi_1 + W\right)^2}{1+\left(V\tan\psi_1 + W\right)^2}\right]^{\frac{1}{2}}$$

$$= \sin^{-1}\left[\frac{\left(\frac{r}{a}\right)^2 U^2(\psi_1)}{1+U^2(\psi_1)}\right]^{\frac{1}{2}}$$

where

$$a = k\,S_1$$

$$V = \left[\frac{A''}{a}\right]\cos\alpha - 1$$

$$W = \left[\frac{A''}{a}\right]\sin\alpha$$

$$\alpha = S_1\hat{k}\,P$$

This equation can be solved numerically.

A symmetrical case about "Y" axis Figure 12 show the use of this method, we can have a value of σ_1 approximately constant if $X = Y + A''$. The values of σ_1, σ_2 at any time "t" for this case are shown in Figure 13.

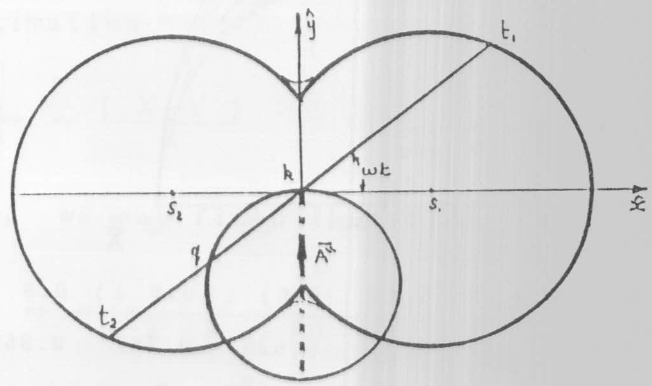

Fig. 12

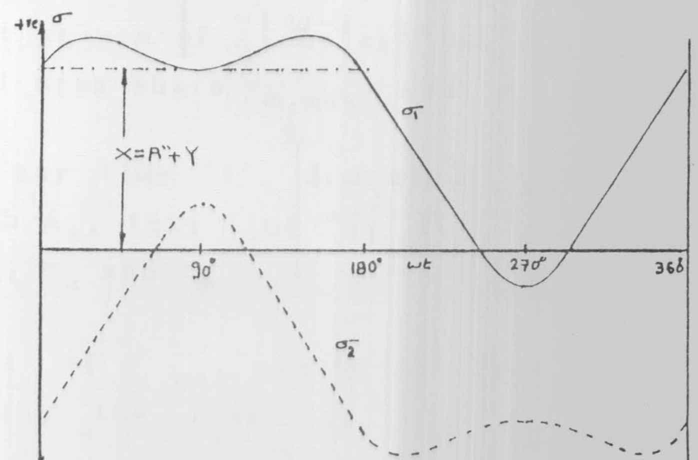

Fig. 13

5- CONCLUSION

The present work is concerned with the combined out of phase loads using a graphical method. In general, it has the advantage of simplyfing the problem, also it give a good visibility for the variation of the direction and magnitude of both shear and normal maximum stresses overall the load cycle. Also the given applications show the capability of this graphical method to deal with special cases and to estimate the complete analysis of the problem.

The simplicity of this method is the possibility of describing the loading condition by two circles and one point, whose values and location depend on the amplitude and phase angle of the applied stresses (A_x, A_y, B, γ, ϕ), and drawing two straight line. The modified method can be used to describe the stress condition by drawing one straight line.

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APPENDIX I

From Figure 14 we have

$$\sigma_1 = \sqrt{r^2 - a^2 \sin^2 \psi} + a \cos \psi + \Lambda'' \cos(180 - \psi - \alpha)$$

where

$$a = k\,S_1 = X - r$$

for $\frac{\partial \sigma_1}{\partial \psi} = 0$

if $V = \frac{\Lambda''}{a} \cos \alpha - 1$

$$W = \frac{\Lambda''}{a} \sin \alpha$$

we get,

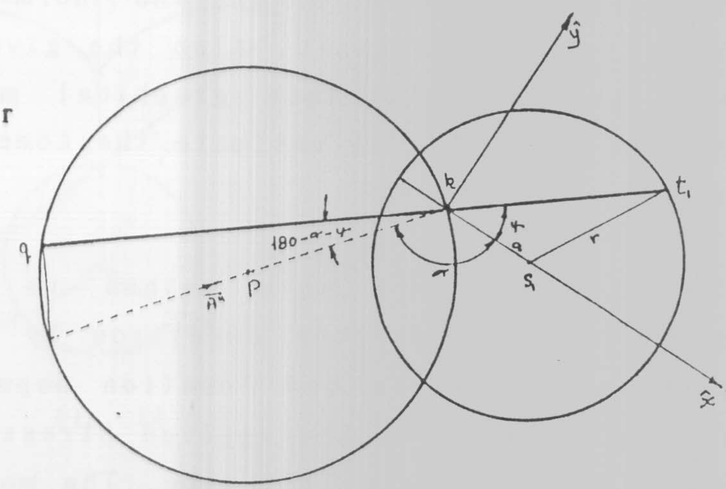

Fig. 14: ($\sigma_1 = q\,t_1$, $\alpha = p\hat{k}S_1$)

$$\frac{\sin \psi_1 \cos \psi_1}{\left[\left(\frac{r}{a} \right)^2 - \sin^2 \psi_1 \right]^{\frac{1}{2}}} = V \sin \psi_1 + W \cos \psi_1$$

if $U(\psi_1) = V \tan \psi_1 + W$ then we fined that

$$\sin^2 \psi_1 = \frac{\left(\frac{r}{a} \right)^2 U^2(\psi_1)}{1 + U^2(\psi_1)}$$