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# A FINITE ELEMENT MODEL FOR THERMAL STRATIFICATION IN STAGNANT LAKES

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**Abstract**

A finite element model for thermal stratification in stagnant lakes or reservoirs is introduced. The time dependent vertical temperature distribution in a deep lake during the yearly cycle of solar heating and cooling is obtained. The numerical model is based on the combined use of the Laplace transformation and the finite element method. In this model the time terms are removed using the Laplace transformation then the associated steady equation is solved by the finite element method in conjunction with the classical Galerkin procedures. The associated temperature is inverted by an accurate numerical method. Two numerical examples are illustrated, the first corresponding to actual lake data and the second corresponding to a laboratory simulation case. In both cases results are in good agreement with the solution obtained by other authors.

## 1. Introduction

Large bodies of water such as lakes provide a convenient source of cooling water supply to electrical generating power plants. The cold water available at depth in lakes is used in the stream condensers and then returned back to the lake. A knowledge of the temperature structure within a large body of water and the changes that take place in it is essential before the perturbation effects of the added heat load on the lake temperature can be assessed.

Lakes in temperature climates show a continuously varying thermal structure throughout the year. Many lakes exhibit an isothermal state in the spring in which a uniform temperature exists throughout the depth. As the season progresses, a vertical temperature profile develops as the water near the surface absorbs solar energy. Later a zone of uniform temperature begins to develop at the surface, falling sharply and then asymptotically to an unchanged temperature deep down. In the autumn the surface temperature begins to fall and the lake again reaches an isothermal condition at the beginning of the winter season.

Theories for the time dependent vertical temperature distribution in a deep lake during the yearly cycle of solar heating and cooling were developed by Dake and Harleman [1]. They assumed that a portion of the incoming solar radiation is to be absorbed at the water surface, whereas the remainder is absorbed exponentially beneath the surface. Heat is also conducted downwards by molecular diffusion. The boundary condition is formulated from a heat flux balance at the water surface, which accounts for back radiation and evaporative heat loss. The solution of the second-order heat equation was obtained by superposition of distinct solutions for the temperature distribution due to effective radiation absorbed at the surface, and for the temperature distribution due to virtually absorbed radiation. They considered some special cases of simple time dependent functions for the incoming radiation and the surface heat loss. Girgis and Smith [2], used the method of variation of parameters for the solution of the heat equation. They have taken into account general time dependent functions for insolation and surface heat loss. As in Dake and Harleman [1], they assumed an exponentially decaying heat source distribution caused by absorbed radiation. However, Snider and Viskanta [4] have improved the understanding of internal energy transfer processes in stagnant water. They considered a plane layer of water of finite depth and analysis of radiative energy transfer within the water was presented. The internal radiant energy absorption rate was not a simple form and they applied an explicit finite difference method for solution. The purpose of this paper is to present a general finite element solution for the transient temperature distribution through a body of stagnant water. The numerical model is based on a hybrid Laplace transformation finite element method, Tamma and Railkar [5]. The computational method can be summarized as follows: after discretizing the spatial domain into finite elements, the Laplace transform is applied to the partial differential equation and boundary conditions. The elements stiffness equations are constructed using Galerkin method and then embedded into the global stiffness matrix and forcing vector. Solution is obtained for the algebric system of equations in the transform domain. The time temperature distribution is calculated using a numerical method for the inversion of Laplace transformation.

## 2. Mathematical Formulation

Consider a lake, reservoir or a pond of such large extent that the energy transfer is essentially one- dimensional. Any inflows and outflows into the body of water are neglected. The water is assumed to be heated by solar radiation only. Within the range of temperature encountered, the variation of density, specific heat and thermal conductivity can be neglected. Under the foregoing assumptions and neglection of eddy diffusivity, the unsteady energy equation governing the temperature distribution becomes

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial z^2}\right) + \frac{H}{\rho c} \qquad (1)$$

where T = temperature,

t = time,

z = vertical coordinate measured downwards,

H = rate of heat generated per unit volume by internal absorption of solar radiation,

$\alpha$ = thermal diffusivity,

$\rho$ = density of the fluid,

c = specific heat of the fluid.

The initial condition is

$$T(z,0) = T_o(z) \qquad (2)$$

The first boundary condition, on the surface z = 0 , requires that the heat generated at the water surface must be equal to the sum of heat conducted into the water and the net heat radiated back to the atmosphere, thus

$$\beta \varphi_o (t) = - \rho c \alpha \left. \frac{\partial T}{\partial z} \right|_{z=0} + \varphi_L (t) \qquad (3)$$

where $\varphi_o$ is the net solar radiation per unit area reaching the water surface, $\beta$ is the proportion of $\varphi_o$ absorbed at the surface and $\varphi_L$ is the total rate of heat loss to the atmosphere. The surface boundary condition (3) can be rewritten as

$$\left. \frac{\partial T}{\partial z} \right|_{z=0} = \frac{-1}{\rho c \alpha} (\beta \varphi_o - \varphi_L) \qquad (4)$$

The second boundary condition is the specification of the heat flux or the temperature at the bottom, i.e.

$$\left. \frac{\partial T}{\partial z} \right|_{z=h} = F(t) \qquad (5.a)$$

or

$$T(h,t) = T_b(t) \qquad (5.b)$$

The solution of (1) subject to (2), (4) and either (5.a) or (5.b) will be provided using the proposed hybrid Laplace transformation finite element algorithm.

## 3. Finite Element Model

The Laplace transformation of the governing equations yields the following equations.

$$S\bar{T}-T_o = \alpha \frac{d^2\bar{T}}{dz^2} + \frac{\bar{H}}{\rho c} \tag{6}$$

subject to

$$\left.\frac{d\bar{T}}{dz}\right|_{z=0} = \frac{-1}{\rho c \alpha} (\beta \bar{\varphi}_o - \bar{\varphi}_L) \tag{7}$$

and

$$\left.\frac{d\bar{T}}{dz}\right|_{z=h} = \bar{F} \tag{8.a}$$

or

$$\bar{T}(h,s) = \bar{T}_b \tag{8.b}$$

where $\bar{T}, \bar{H}, \bar{\varphi}_o, \bar{\varphi}_L, \bar{F}$ and $\bar{T}_b$ are the Laplace transforms of the temperature $T$, the rate of heat generation by internal absorption $H$, the rate of solar radiation $\varphi_o$, the rate of heat loss $\varphi_L$, the heat flux at the bottom $F$ and the temperature at the bottom $T_b$ respectively. Here we have used the symbol for ordinary rather than partial differentiation because S is only a parameter in the problem, no differentiation with respect to S is involved.

Let us first consider the case of known heat flux at the bottom, i.e. condition (8.a) is imposed at z=h. Applying the Galerkin procedures, the weak solution is obtained from

$$\int_0^h \left[\frac{d^2\bar{T}}{dz^2} - \frac{s}{\alpha}\bar{T} + \frac{\bar{H}}{\rho c \alpha} + \frac{T_o}{\alpha}\right]\eta_i \, dz$$

$$+ \left[\frac{d\bar{T}}{dz} + \frac{1}{\rho c \alpha}(\beta\,\bar{\varphi}_o - \bar{\varphi}_L)\right]\xi_i \Bigg|_{z=0}$$

$$+ \left(\frac{d\bar{T}}{dz} - F\right)\xi_i \Bigg|_{z=h} = 0 \qquad (9)$$

where $\eta_i$, $\xi_i$ are sets of interior and boundary test functions. Green's theorem is applied for the first term in (9) and the boundary test functions are choosen such that $\xi_i = -\eta_i$ . The domain [0,h] is partitioned into (n-1) line elements of length $l^e$, then the finite element approximation of T is given by

$$\bar{T} = \sum_{j=1}^{n} N_j(z)\,\bar{T}_j, \qquad (10)$$

where $\bar{T}_j$ are the unknown nodal temperatures in the transform domain, $N_j(z)$ are the shape (trial) functions and n is the number of nodes.

If we consider the test space is equal to the trial space, then the weak solution is obatined from

$$\int_0^h \sum_{j=1}^{n} \bar{T}_j \left(\frac{dN_i}{dz}\,\frac{dN_j}{dz} + \frac{s}{\alpha} N_i\, N_j\right)dz$$

$$= \frac{1}{\rho c \alpha} \int_0^h (\bar{H} + \rho c T_o) N_i \, dz$$

$$- \frac{1}{\rho c \alpha} (\beta \bar{\varphi}_o - \bar{\varphi}_i) N_i \Big|_{z=0} + \bar{F} N_i \Big|_{z=h} \qquad (11)$$

$$i = 1, \ldots n$$

In a matrix form, equation (11) is written as

$$A\bar{T} = f, \qquad (12)$$

where A is the system stiffness matrix, $\bar{T}$ is the vector of the unknown nodal temperatures and f is the vector of the known forcing terms.

The integrals over the region [0,h] are performed as the sum of integrals over the individual elements, i.e

$$A = \sum_e A^e, \qquad (13)$$

where $A^e$ is called the element matrix and

$$f = \sum_e f^e, \qquad (14)$$

where $f^e$ is the element force vector. Typical finite element generalized formulation are thus obtained as

$$A = \sum_e \int_{l^e} \left\{ [B^e]^T [B^e] + \frac{s}{\rho} [N^e]^T [N^e] \right\} dz. \qquad (15)$$

$$f = \sum \frac{1}{\rho c \alpha} \int_{l^e} (\bar{H} + \rho c T_o) [N^e]^T dz$$

$$- \frac{1}{\rho c \alpha} (\beta \bar{\varphi}_o - \bar{\varphi}_L) [N^e]^T_{z=0} + F[N^e]^T_{z=h} , \qquad (16)$$

where $[N^e]$ and $[B^e]$ are the element temperature and temperature gradient interpolation functions matrices.

For the case of known temperature at the bottom (z=h), the last term in (16) is eliminated and the Dirichlet condition (8.b) is imposed after the assembly is carried out.

## 4. Solution Method

Solution of the linear system (12) is performed using direct Gaussian elimination. The numerical inversion is carried out by the method of Honigs and Hirdes [3].

The Laplace transform of a real function g(t), g(t)=0 for t < 0, and its inversion formula are defined as

$$G(s) = L[g(t)] = \int_o^{\infty} e^{-st} g(t)\, dt \qquad (17)$$

and

$$g(t) = L^{-1} [G(s)] = \frac{1}{2\pi i} \int_{v-i\infty}^{v+i\infty} e^{st} G(s)\, ds \qquad (18)$$

with $s=v+iw$ and $v,w$ are real. $v$ is arbitrarily chosen to be greater than the real part of the dominating sigularity of $G(s)$. The integrals in (17) and (18) exist for $Re(s) > a \in R$ if

(a) $g$ is locally integrable,

(b) there exist $t_o \geq 0$ and $\kappa, a \in R$ such that $|g(t)| \leq \kappa e^{at}$ for all $t \geq t_o$,

(c) for all $t \in (0, \infty)$ there is a neighbourhood in which $g$ is of bounded variation.

The possibility to choose $v > 0$ arbitrarily is the basis of the method of Durbin [3]. This method is now described: if we substitute $s = v + iw$ in (17), we can write

$$G(s) = \int_0^\infty e^{-vt} g(t)\,[\cos wt - i \sin wt]\,dt, \qquad (19)$$

or

$$G(s) = Re\,[G(s)] + i\,Im\,[G(s)], \qquad (19.a)$$

then the inversion formula is

$$g(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{vt}\{\cos wt + i \sin wt\}\{Re[G(s)] + i Im[G(s)]\}\, i\, dw$$

or we can write

$$g(t) = \frac{e^{vt}}{2\pi}\Big[\int_{-\infty}^{\infty}\{ Re[G(s)]\cos wt - Im[G(S)]\sin wt\}\, dw + i\int_{-\infty}^{\infty}\{ Im[G(s)]\cos wt + Re[G(S)]\sin wt\}\, dw\Big]. \qquad (20)$$

Substituting (19) and (19.a) in (20) and using (17) we get

$$g(t) = \frac{e^{vt}}{\pi} \int_{0}^{\infty} \{ Re[G(s)]\cos wt - Im[G(s)]\sin wt\} \, dw. \qquad (21)$$

A Fourier series expansion of $g(t)e^{-vt}$ in the interval $[0,2\tau]$ yields the Durbin's approximate formula

$$g(t) = \frac{e^{vt}}{\tau} \{ \frac{-1}{2} Re [G(v)] + \sum_{0}^{\infty} Re [G(v+i \frac{k\pi}{\tau} )] \cos (\frac{k\pi}{\tau} t)$$

$$- \sum_{0}^{\infty} Im[G(v+i \frac{k\pi}{\tau})] \sin \frac{k\pi}{\tau} t$$

$$- E(v,t,\tau), \qquad (22)$$

where $E(v,t,\tau)$ is the discretization error given by

$$E(v,t,\tau) = \sum_{1}^{\infty} e^{-2vk\tau} g(2k\tau + t)$$

The corrector method allows a reduction of the discretization error without enlarging the truncation error. Three different methods for the acceleration of convergence are described in Honig and Hirdes [3], namely the $\varepsilon$ - algorithm, the min-max method and a method based on curve fitting. As the infinite series in (22) can only be summed up to a finite number N of terms, then a good choice of the free parameters N and $CON = v\tau$ is not only important for the accuracy of the results but also for the application of the corrector method and the methods for the acceleration of convergence. These methods do not improve the results if the parameters are chosen badly. Two method which approximately determine the optimal $v$ for fixed N and $\tau$ are presented in Honig and Hirdes [3].

## 5. Numerical Examples

Two sets of observation are available in the literature: the first, occuring in nature, consists of temperature observed in Lake Tahoe, and the second is a controlled experiment using radiant heat producing lamps in a laboratory Girgis and Smith [2]. In both cases the heat source term due to internal absorption of solar radiation is

$$H = -\left(\frac{\partial \varphi}{\partial z}\right) = \zeta\,(1-\beta)\,\varphi_o\, e^{-\zeta z}$$

In the case of Lake Tahoe, the values of the parameters are:
$\varphi_o = 6.5*10^6$ cal/mt$^2$ day, $\zeta = 0.05$ mt$^{-1}$, $\beta = 0.4$, $\varphi_L = 3*10^6$ cal/mt$^2$

$\rho = 1.0$ gm/cm$^3$, c=1.0 cal/gm. $^o$C and $\alpha = 0.0014$ cm$^2$/sec.
The initial distribution is assumed to be $T_o = 4^o$C. The calculated temperature distributions for t = 40, 80 and 120 days are shown in Figure (1). The results are in good agreement with those in Gigis and Smith [2].

In the laboratory experiment, the parameters are:
$\varphi_o = 0.01$ cal/cm$^2$. sec, the loss at the surface is assumed to be proportional to $t^{\frac{1}{2}}$ such that $\varphi_L = \beta\varphi_o\Omega t^{\frac{1}{2}}$, $\beta = 0.75$, $\zeta = 0.01$ cm$^{-1}$ h=75 cm, $\Omega = 0.004$ sec$^{-1}$, $T_o = 21\,^o$C $= T_b$. The results are plotted in Figure (2), together with results in Girgis and Smith [2].

## 6. Conclusion

The combined use of Laplace transformation and the finite element method is a powerful method of analysis. There is no time step, hence

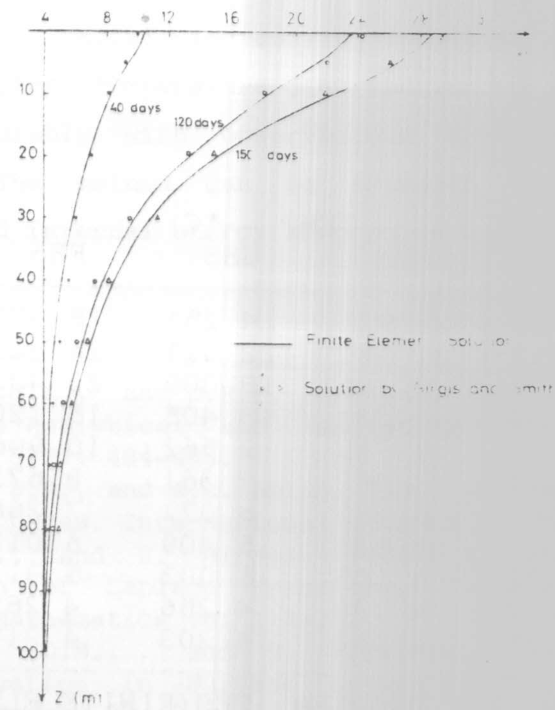


*Figure (1): Finite Element Solution For Temperature Distribution in Lake Tahoe.*

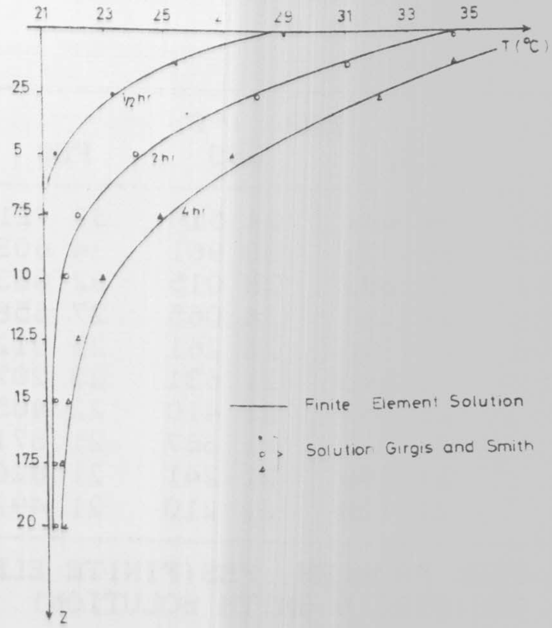


*Figure (2) : Finite Element Solution For Temperature Distribution in Laboratory.*

| TIME (day) | 40 | | 120 | | 150 | |
|---|---|---|---|---|---|---|
| Z (mt) | TEMP (°C) FES | GSS | TEMP (°C) FES | GSS | TEMP. (°C) FES | GSS |
| 0 | 10.233 | 10.027 | 23.921 | 24.251 | 28.921 | 29.481 |
| 5 | 9.189 | 9.211 | 21.561 | 22.011 | 26.326 | 26.101 |
| 10 | 8.316 | 8.209 | 18.106 | 18.009 | 22.402 | 22.009 |
| 20 | 6.901 | 7.033 | 13.231 | 13.405 | 15.120 | 15.056 |
| 30 | 5.863 | 6.101 | 9.782 | 9.562 | 10.906 | 11.264 |
| 40 | 5.210 | 5.391 | 7.492 | 7.361 | 8.671 | 8.492 |
| 50 | 4.887 | 5.001 | 6.285 | 6.072 | 7.268 | 7.106 |
| 60 | 4.531 | 4.727 | 5.521 | 5.409 | 6.012 | 5.921 |
| 70 | 4.318 | 4.403 | 5.107 | 5.003 | 5.311 | 5.203 |
| 80 | 4.120 | 4.237 | 4.190 | 4.256 | 4.263 | 4.296 |
| 90 | 4.086 | 4.128 | 4.163 | 4.203 | 4.217 | 4.259 |

**TABLE(1): RESULTS FOR LAKE PROBLEM. FES(FINITE ELEMENT SOLUTION) GSS(GIRGIS-SMITH SOLUTION)**

| TIME (hr) | 0.5 | | 2.0 | | 4.0 | |
|---|---|---|---|---|---|---|
| Z (cm) | TEMP (°C) FES | GSS | TEMP (°C) FES | GSS | TEMP. (°C) FES | GSS |
| 0.00 | 28.925 | 29.100 | 34.461 | 34.510 | 37.721 | 37.341 |
| 1.25 | 25.396 | 25.501 | 30.721 | 30.961 | 34.603 | 34.572 |
| 2.50 | 23.418 | 23.381 | 27.892 | 28.015 | 32.083 | 32.127 |
| 5.00 | 21.511 | 21.408 | 24.211 | 24.065 | 27.658 | 27.431 |
| 7.50 | 21.000 | 21.000 | 22.396 | 22.261 | 25.012 | 24.903 |
| 10.00 | | | 21.540 | 21.631 | 23.207 | 22.987 |
| 12.50 | | | 21.467 | 21.410 | 22.405 | 22.231 |
| 15.00 | | | 21.311 | 21.367 | 21.871 | 21.726 |
| 17.50 | | | 21.199 | 21.241 | 21.626 | 21.503 |
| 20.00 | | | 21.108 | 21.210 | 21.492 | 21.501 |

**TABLE(2): RESULTS FOR TANK PROBLEM. FES(FINITE ELEMENT SOLUTION). GSS(GIRGIS-SMITH SOLUTION).**

the method is useful in solving long-time problems. The numerical results for the temperature distribution in a stagnant lake or tank compare favourably with observations obtained in nature and the laboratory. The method can be extended to more general boundary conditions and internal energy absorption rate.

## 7. References

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