Alex. Eng. J., Alex. Univ., Volume, 28 No.2, pp. 511 - 531 (1989)

ON THE APPROXIMATE PERIODIC SOLUTION OF AN ELECTRONIC-TUBE OSCILLATOR

Mohamed Hassan Eid

Assoc. Prof. Department of Engineering Mathematics,

Faculty of Engineering, Alexandria University

Alexandria, Egypt

Abstract

We consider the operation of an electronic-tube circuit of the standard (induction coupled) type. The equation of the circuit can be transformed-via certain relations - to Van der Pol equation. This equation is related to the nonlinear auto-oscillations, therefore, in the present paper, the method of Myshkis has been discussed in details and used to solve it to the third order accuracy. Also, the theory of synchronisation and the formula of Airy in the second approximation have been applied to it. Comparisons with other methods are discussed.

1. Introduction

In recent years, great interest has been given to the investigation of oscillatory processes arising in many engineering systems such as electrical and radio engineering, vibrotechnics, celectial mechanics, instrument making, and control.

From the theory of differential equations, the autonomous systems appear merely as a special case of more general nonautonomous ones. But in the theory of oscillations, the studies started from autonomous cases, inasmuch as the Van der Pol equation (an electronic-tube oscillator), which appeared to be the first work on the autonomous cases.

For autonomous systems, one can always replace t (time) by $(t + t_o)$ where t_o is an arbitrary constant (the phase), and still have a solution of the same equation. This means that one can select arbitrarily the time origin. This permits the selection of time-origin at the instant when velocity is zero. The period of oscillation, if it is known, depends on the parameters of the system. In other words, it is determined by the differential equation itself [1].

Recalling that limit cycles represent the stationary states of oscillations (the stationary motion is independent of initial conditions) [1], the motion described by a stable limit cycle is called an auto-oscillation. In addition, the oscillations of limit cycle have an important property of self-starting or of self excitation as in the example of an electronic-tube oscillator. Many authors create and develop different mehods including functional-analytic, numerical-analytic and numerical methods to investigate periodic solutions of differential equations [1-7]. The application of the trigonometric-collocation method to construct the periodical solutions to autonomous systems of equation creates additional difficulties: namely :(i) the period of their solutions is unknown and (ii) the solutions themselves are not isolated [8]. These difficulties cause substantial obstacles in extending both methods (Numerical-Analytical Method and Collocation Method) to autonomous systems of equations [9-11].

In the present paper, the method of Myshkis [12], the theory of synchronisation [13 (1,2,3), 15(1,2)] and the formula of Airy in the second approximation [13(3)] are to be adopted for auto-oscillation system, in studying the periodic solution of Van der Pol equation in the third approximation as an example, for making comparisons with other methods.

The fundamental goal, however, which one sets himself is the presentation of practical procedures for computing the auto-oscillations.

2. Construction Of The Periodic Solution Of Auto-Oscillations Systems

We proceed by investigating the following differential equation [12].

$$\ddot{x} + \omega_o^2 x = \lambda f_1 (x,\dot{x}) + \lambda^2 f_2 (x,\dot{x}) + \dots , \quad (\cdot = \frac{d}{dt}) \tag{1}$$

in which "λ" is a small parameter. Eq. (1) describes an autonomous system with one degree of freedom. Such a system may have limit cycles corresponding to auto-oscillations. Here, one will show how they can be determined.

As essential distinction between forced oscillations and auto-oscillations is that the frequency "ω" of an auto-oscillation is not known beforehand but is found in the calculation process. Therefore, one takes the expansion

$$\omega = \omega_o + \omega_1 \lambda + \omega_2 \lambda^2 + \dots \tag{2}$$

in which all the coefficients except ω_o are undetermined. If we introduce the new time scale $\theta = \omega t$, thus, Eq. (1) is rewritten in the form

$$(\omega_o + \omega_1\lambda + \omega_2\lambda^2 + \dots)^2 \ddot{x} + \omega_o^2 x = \lambda f_1 \left(x, (\omega_o + \omega_1\lambda + \omega_2\lambda^2 + \dots)\dot{x}\right) + \dots \tag{3}$$

where the dot is now used for designating differentiation with respect to the new time scale "θ". As before, the solution is looked for in the form of an expansion in powers of λ : that is

$$x = x_o(\theta) + x_1(\theta)\lambda + x_2(\theta)\lambda^2 + \dots \tag{4}$$

The substitution of Eq. (4) leads to the following relations:

$$\left.\begin{aligned} &\ddot{x}_o + x_o = 0 \\ &\ddot{x}_1 + x_1 = -\frac{2\omega_1}{\omega_o}\ddot{x}_o(\theta) + \frac{1}{\omega_o^2} f_1(x_o(\theta), \omega_o \dot{x}_o(\theta)) \end{aligned}\right\} \tag{5}$$

It may be interesting to note here then another peculiarity of autonomous vibrations is that the initial phase can be chosen in an arbitrary way. This means that any point of the cycle in question can be taken as initial point corresponding to the instant t=0 (this is not the case for forced oscillations). In particular we can put $x(0)=0$

$$x_o(0) = 0,\ x_1(0) = 0,\ x_2(0) = 0\ ,\ \ldots \tag{6}$$

Now, we can proceed to determine, in succession, the solution of Eqs. (5). From the first equation (5), taking into account the initial conditions (6), we obtain the expression

$$x_o = b \sin \theta \tag{7}$$

where the amplitude "b" is yet unknown. Substituting Eq. (7) into the right-hand side of the second equation (5) and eliminating the secular terms, we obtain the equalities

$$\left.\begin{aligned} &\int_0^{2\pi} [2\,\omega_o\,\omega_1\, b \sin\theta + f_1\,(b \sin\theta\,,\, \omega_o\, b \cos\theta\,)] \cos\theta\, d\theta = 0 \\ &\text{and} \\ &\int_0^{2\pi} [2\,\omega_o\,\omega_1\, b \sin\theta + f_1\,(b \sin\theta\,,\, \omega_o\, b \cos\theta\,)] \sin\theta\, d\,\theta = 0 \end{aligned}\right\} \tag{8a}$$

that can also be rewritten in the forms :

$$\left.\begin{aligned} &\int_0^{2\pi} f_1\,(b \sin\theta\,,\, \omega_o b \cos\theta\,) \cos\theta\, d\theta = 0 \\ &\omega_1 = -\frac{1}{2\pi b\omega_o} \int_0^{2} f_1\,(b \sin\theta\,,\, \omega_o\, b \cos\theta\,) \sin\theta\ d\,\theta \end{aligned}\right\} \tag{8b}$$

The first equality (8a) determines the amplitude "b". After "b" has been found, the second equality (8a) yields an explicit expression for "ω_1". Then, we take the second equation (5) and from it the expression

$$x_1 = \varphi_1(\theta) + b_1 \sin\theta \qquad (\varphi_1(0) = 0).$$

The condition of the absence of resonant oscillations for the subsequent equation results in two determining equations from which b_1 and " ω_2" are found and so on.

We remark that conditions (8a) may be reversed according to initial conditions as in Section 4.

To test the cycle thus constructed for stability, one must rewrite Eq. (1) in the form of first-order equations and then apply the general condition estabilished by H. Poincare [13] which shows that if

$$\lambda \int_0^{2\pi} f'_{\dot{x}} (b \sin\theta , \omega_o b \cos\theta) \, d\theta < 0 , \quad (' = \frac{\partial}{\partial \dot{x}}) \tag{9}$$

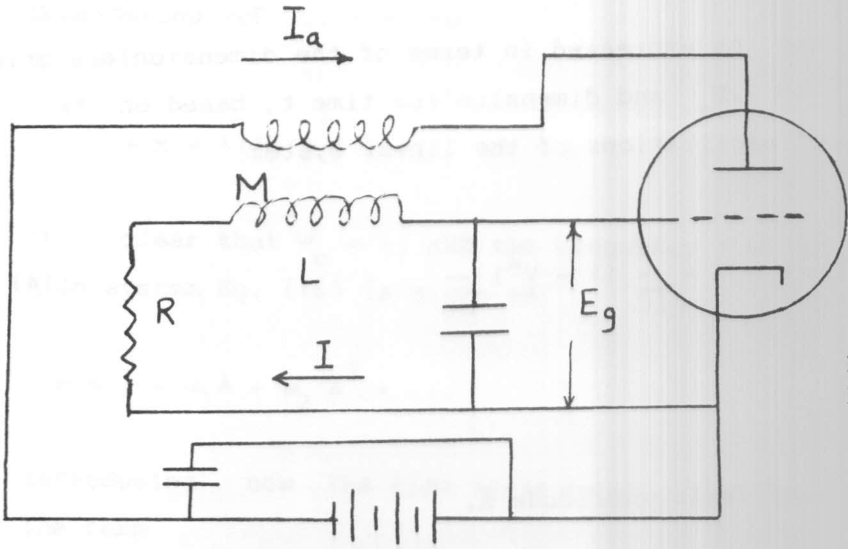
the cycle is stable and if the left member of (9) is positive it is unstable.

3. The Circuit Equation of Van Der Pol

Let us consider a standard electronic-tube circuit [14] with inductive coupling (see Fig. 1). With usual notation, the differential equation of the oscillatory circuit is:

$$L \frac{dI}{d\tau} + RI + \frac{Q}{C} = M \frac{dI_a}{d\tau} \tag{10}$$

the right-hand term in this differential equation indicates the action of the anode current I_a exerted inductively, M being the coefficient of mutual inductance between the plate and the grid circuits.



Fug.
Fig. (1)

Here, τ is the physical time (sec), and the mutual inductance M is positive. The windings are arranged to oppose the voltage drop across L. The grid current is assumed to be negligibily small. The plate current I_a depends mainly on the grid voltage E_g and is given by the characteristic of the vacuum tube. An analytic form which approximates tube characteristic well for $E_g/E_s < 1$ (E_s = characteristic saturation voltage) is

$$I_a = \sigma \left(E_g - \frac{1}{3} \frac{E_g^3}{E_s^2}\right) \tag{11}$$

where σ is the tube conductance (mho).

Now, $Q(\tau)$ is the charge on the capacitor, so that

$$E(\tau) = \frac{Q}{C}, \tag{12}$$

$$I = \frac{dQ}{d\tau} = C\frac{dE}{d\tau} \tag{13}$$

Thus, Eq. (10) can be expressed in terms of the dimensionless grid voltage $V = E_g(\tau)/E_s$ and dimensionless time t, based on the natural frequency of oscillations of the linear system

$$\frac{d^2V}{dt^2} + R\sqrt{\frac{C}{L}}\frac{dV}{dt} + V = \frac{M\sigma}{\sqrt{LC}}(1 - V^2)\frac{dV}{dt} \tag{14}$$

$$t = \omega\tau, \quad \omega = \frac{1}{\sqrt{LC}}$$

By choosing a characteristic amplitude A,

$$V(t) = A\,x(t)$$

Eq. (14) can be brought to the form

$$\frac{d^2x}{dt^2} - \lambda(1-x^2)\frac{dx}{dt} + x = 0 \tag{15}$$

where

$$\lambda = \frac{M\sigma}{\sqrt{LC}} - R\sqrt{\frac{C}{L}} = \frac{M\sigma A^2}{\sqrt{LC}}, \quad A = \sqrt{1 - \frac{RC}{M\sigma}}$$

This form is suitable for studying weak nonlinear effects.

4. Application Of The Method

In this Section, we present an aplication of the method constructed in

Sec. 2 to the Van der Pol Equation (15). The main reason for such a choice is the comparison with other previous methods to establish the validity of the present paper method.

Considering VDP Eq. (15) as

$$\ddot{x} + x = \lambda(1 - x^2)\,\dot{x}\,, \quad 0 < \lambda < 1 \qquad \left(. = \frac{d}{dt} \right) \tag{16}$$

It is clear that $\omega_o = 1$, and the frequency " ω" of the auto-oscillation system Eq. (16) is given as

$$\omega = 1 + \omega_1 \lambda + \omega_2 \lambda^2 + \ldots \tag{17}$$

Introducing now the time scale $\theta = \omega t$, then Eq. (16) is rewritten in the form

$$(1 + \omega_1 \lambda + \omega_2 \lambda^2 + \ldots)^2 \ddot{x} + x = \lambda(1 - x^2)(1 + \omega_1 \lambda + \omega_2 \lambda^2 + \ldots)\,\dot{x} \tag{18}$$

where $\left(. = \frac{d}{d\theta} \right)$

Now the solution takes the form

$$x = x_o(\theta) + x_1(\theta)\lambda + x_2(\theta)\lambda^2 + \ldots \tag{19}$$

The substitution of Eq. (19) into Eq. (18) leads to the relations

$$\ddot{x}_o + x_o = 0 \tag{20}$$

$$\ddot{x}_1 + x_1 = -\,2\,\omega_o\,\ddot{x}_o + \dot{x}_o\,(1 - x_o^2) \tag{21}$$

$$\ddot{x}_2 + x_2 = -2\omega_1\ddot{x}_1 - (\omega_1^2 + 2\omega_2)\ddot{x}_0 + \dot{x}_1 + \omega_1\dot{x}_0 - [x_0^2(\dot{x}_1 + \omega_1\dot{x}_0) + 2x_0\dot{x}_0 x_1] \tag{22}$$

$$\ddot{x}_3 + x_3 = -2\omega_1\ddot{x}_2 - \ddot{x}_1(\omega_1^2 + 2\omega_2) - \ddot{x}_0(2\omega_3 + 2\omega_1\omega_2) + \dot{x}_2 + \omega_1\dot{x}_1 + \omega_2\dot{x}_0$$

$$- [x_0^2(\dot{x}_2 + \omega_1\dot{x}_1 + \omega_2\dot{x}_0) + 2x_0 x_1(\dot{x}_1 + \omega_1\dot{x}_0) + \dot{x}_0(x_1^2 + 2x_0 x_2)] \tag{23}$$

....

The initial phase can be chosen as x(0) = 0, then

$$x_0(0) = 0,\ x_1(0) = 0,\ x_2(0) = 0,\ \ldots$$

Solving Eq. (20) gives

$$x_0 = b\cos\theta \tag{24}$$

where b is found from <u>the first approximation</u>, as follows: Substituting Eq. (24) in Eq. (21) we get

$$\ddot{x}_1 + x_1 = 2\omega_1 b\cos\theta - b\sin\theta + b^3\cos^2\theta\sin\theta \tag{25}$$

Imposing the nonresonance conditions (8a) on Eq. (25), the second equation (8a) gives

$$-\pi b + \frac{\pi}{4}b^3 = 0,$$

whence b = + 2 or b = -2. The solution of Eq. (16) being invariant with respect to the transformation x to -x, both values of b yield the same cycle. Therefore, we can take any one of them, say b = 2.

Now applying stability criterion (9) which gives the value

$$\int_{o}^{2\pi} (1 - 4 \sin^2\theta)\, d\theta = - 2\lambda\pi < 0 .$$

We conclude that the constructed cycle is stable for $\lambda > 0$. From the first equation (8a) and Eq. (25) we find

$$2 \pi b \omega_1 = 0$$

which gives $\omega_1 = 0$, this means that to analyse the variation of the frequency "ω" it is necessary to use the subsequent terms of the expansion. (The equality $\omega_1 = 0$ is a direct consquence of the fact that Eq. (16) goes onto itself under the transformation λ to $-\lambda$ and t to -t because this implies that "ω" is an even function of the parameter λ, and therefore its expansion in powers of λ can only involve even powers). The computations emphasize this fact as it follows:

Considering $b = 2$, $\omega_1 = 0$ Eqs. (24) and (25) become respectively

$$x_o = 2 \cos\theta \tag{24a}$$

$$\ddot{x}_1 + x_1 = 2 \sin 3\theta \tag{26}$$

its solution is

$$x_1 = \frac{3}{4} \sin\theta - \frac{1}{4} \sin 3\theta + b_1 \cos\theta \tag{27}$$

where b is obtained in <u>the second approximation</u> as follows:

$$\ddot{x}_2 + x_2 = (4\,\omega_2 + \tfrac{3}{4})\cos\theta + b_1(-\sin\theta + 12\cos^2\theta\sin\theta) - \tfrac{3}{4}\cos 3\theta +$$

$$+ 3\cos^2\theta\cos 3\theta - 3\cos^3\theta - 2\cos\theta\sin 3\theta\sin\theta +$$

$$+ 6\sin^2\theta\cos\theta \tag{28}$$

Applying the conditions (8a) guaranteeing the absence of resonant terms in x(t) on Eq. (28), we find:

$$b_1 = 0 \tag{28a}$$

$$\omega_2 = -\frac{1}{16} \tag{28b}$$

Owing to Eqs. (28a,b) ; Eqs. (27) and (28) become respectively

$$x_1 = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta \tag{27a}$$

$$\ddot{x}_2 + x_2 = -\frac{3}{2}\cos 3\theta + \frac{5}{4}\cos 5\theta \tag{29}$$

Solution of Eq. (29) will be

$$x_2 = \frac{3}{16}\cos 3\theta - \frac{5}{96}\cos 5\theta + b_2\cos\theta \tag{30}$$

where b_2 is given from the third approximation as follows:

Substituting Eq. (30) in Eq. (23) we get,

$$\ddot{x}_3 + x_3 = 4\omega_3 \cos\theta + \frac{1}{32}\sin\theta + \frac{27}{32}\sin 3\theta - \frac{25}{96}\sin 5\theta$$

$$+ b_2 (1+8\cos^2\theta + 2\cos 2\theta)\sin\theta + \frac{9}{8}\cos 2\theta \sin 3\theta$$

$$- \frac{25}{48}\cos 2\theta \sin 5\theta - \frac{11}{4}\sin\theta \cos^2\theta - \frac{3}{4}\cos\theta \sin 3\theta \cos 3\theta$$

$$+ \frac{3}{4}\cos^2\theta \sin 3\theta + \frac{15}{4}\cos\theta \sin\theta \cos 3\theta + \frac{9}{8}\sin^3\theta -$$

$$+ \frac{1}{8}\sin\theta \sin^2\theta\, 3\theta - \frac{5}{12}\sin\theta \cos 5\theta - \frac{3}{4}\sin^2\theta \sin 3\theta \qquad (31)$$

According to the nonresonance conditions (8a) for x_3 (t) we find:

$$b_2 = -\frac{1}{8}; \qquad (31a)$$

$$\omega_3 = 0 \qquad (31b)$$

Noting that we get $\omega_3 = 0$ which emphasize the previous given fact that we have for the frequency " ω" the even powers of λ only i.e.

$$\omega = 1 + \omega_2 \lambda^2 + \omega_4 \lambda^4 + \ldots$$

According to Eq. (31a); Eq. (30) becomes

$$x_2 = -\frac{1}{8}\cos\theta + \frac{3}{16}\cos 3\theta - \frac{5}{96}\cos 5\theta \qquad (30a)$$

Substituting Eqs. (24a), (27a) and (30a) into Eq. (20) we get the solution of Eq. (18) as

$$x = 2\cos\theta + \lambda\left[\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right] + \lambda^2\left[-\frac{1}{8}\cos\theta + \frac{3}{16}\cos 3\theta - \frac{5}{96}\cos 5\theta\right] \qquad (32)$$

with $\theta = \omega t$,

$$\omega = 1 - \frac{1}{16}\lambda^2 + O(\lambda^4) \qquad (33)$$

equation (33) yields to the period of oscillations as

$$T = 2\pi\left(1 + \frac{\lambda^2}{16} + \dots\right)$$

5. The Theory of Synchronisation [15, 16]

If we consider the following differential equation

$$\ddot{x} + \omega_o^2 x = \lambda\omega_o f(x, \dot{x}) \qquad (34)$$

and changing the independent variable by $\varphi_1 = \omega_o t$

$$x = b\cos\varphi_1, \quad \dot{x} = -\omega_o b\cos\varphi_1, \quad \varphi_1 = \omega_o t$$

We have the associated functions for equation (34) as which is the same as obtained in Section 4.

Applying the condition of stability (37), we get $\frac{dY}{db} = -2$

then, the system is stable as in Section 4.

The relative increase φ of the period is small ($\varepsilon = 0$) of second order in λ, it is necessary to use Airy's formula in the second approximation as in Section 6, because the first approximation is valid only for $O(\lambda^2)$.

6. Formula of Airy In Second Approximation [15,16]

We recall Airy's formula here to compute the period of the following equation

$$\ddot{x} + \omega_o^2 x = \lambda f(x, \dot{x}) \tag{41}$$

where the relative increasing of the period of auto-oscillation (41) is given by

$$\left.\begin{aligned} \frac{\Delta T}{T} &= \frac{I}{b} + \frac{J+K}{b} \\ \text{with} \quad I &= \frac{1}{2\pi} \int_o^{2\pi} G(x_1, \dot{x}_1) \cos\varphi \, d\varphi , \\ J &= \frac{1}{2\pi} \int_o^{2\pi} [G(x_1, \dot{x}_1)]^2 \, d\varphi , \end{aligned}\right\} \tag{42}$$

and

$$K = \frac{1}{2\pi} \iint [2\, G(x_1, \dot{x}_1) \cos\varphi - b \frac{\partial}{\partial x} G(x_1, \dot{x}_1)].$$

$$G(b \cos\dot{\varphi}, -\omega_0 b \sin\dot{\varphi}) \sin\dot{\varphi}\, d\varphi\, d\dot{\varphi}, \quad 0 < \dot{\varphi} < \varphi < 2$$

where

$$G(x, \dot{x}) = \frac{\lambda f(x, \dot{x})}{\omega_0^2}$$

and where : $x_1 = b \cos\varphi$, $\dot{x}_1 = -\omega_0 b \sin\varphi$, $\varphi = \omega_0 t$.

Applying this formula (42) to VDP equation (16) and all calculations are made, we have

$$I = 0, \quad J = 2\lambda^2, \quad K = -\frac{7}{4}\lambda^2, \text{ and}$$

$$J + K = \frac{\lambda^2}{4} \qquad (43)$$

Then, substituting Eq. (43) in the first formula (42) putting b = 2, one gets

$$\frac{\Delta T}{T} = \frac{\lambda^2}{16} \quad \text{or} \quad \frac{\omega_0 - \omega}{\omega_0} = \frac{\lambda^2}{16}; \text{ putting } \omega_0 = 1, \text{ one can obtain}$$

$$\omega = 1 - \frac{\lambda^2}{16} \qquad (44)$$

Equation (44) reveals that the results for the frequency previously obtained by equation (33) is the same.

7. Concluding Remarks

Thus, although a Van der Pol oscillator with a small value appears as an almost ideal image of the harmonic oscillator, in reality it contains a germ of a complicated structure consisting of an infinite spectrum of amplitude and phase modulations escapes observation only because this structure is very small if λ is small.

We solved VDP equation (16) applying the method of strained coordinates [19] for periodic solution and we obtained the expansion

$$x = 2\cos\theta + \lambda\left[\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right] + \lambda^2\left[-\frac{13}{96}\cos\theta + \frac{3}{16}\cos 3\theta - \frac{5}{96}\cos 5\theta\right], \quad \theta = \omega t \tag{45}$$

and the expansion of "ω" is in full agreement with equation (33). These results are also obtained by Cunningham [20] using a perturbation method and Chaleat [16 (3)] using the theory of perturbation. Equation (45) coincides with equation (32) except a minor error of the coefficient (-12/96) of $\cos\theta$ in $O(\lambda^2)$ in Eq. (32) and (-13/96) in Eq. (45). Roseau [3] utilizing a perturbation method and Balbi [17] applying an averaging method solved VDP equation (16) and obtained results in full agreement with equations (32) and (33). Nayfeh [6(1)] treated rigorously VDP oscillator Eq. (16) using the Krylov-Boglioubov-Mitropolsky technique which gave

$$x = 2\cos\theta - \frac{\lambda}{4}\sin 3\theta + \lambda^2\left[-\frac{3}{32}\cos 3\theta - \frac{5}{96}\cos 5\theta\right] + O(\lambda^3)$$

then, the method of averaging using Lie series and transforms, the derivative-expansion method, also, both Malkin [18] and Bogolioubov-Mitropolsky [2] obtained the solution as

$$x = 2\cos\theta - \frac{\lambda}{4}\sin 3\theta + O(\lambda^2)$$

All mentioned methods in this paper give the same expansion (33) of the frequency "ω".

Nayfeh [6(2)] showed that neither the Lindstedt-Poincare technique nor the method of renormalization is capable of yielding the transient response for self-excited oscillators such as VDP equation. Then, he showed that the methods of multiple scales and averaging can yield it.

In our notation, Nayfeh [6(2)] and others such as in [1-3,15-21] studied the general form

$$\ddot{x} + x = \lambda f(x,\dot{x})$$

Meanwhile, the method of Myshkis discussed in this paper is fit for a more general form of equation (1) and gives a more accurate results.

References

[1] Minorsky, N., "Nonlinear Oscillations", Krieger Publishing Co., Inc. New York, (1974).

[2] Bogolioubov, N.N., et Mitropolsky, Yu., "Les Méthodes asymptotiques en Théorie des Oscillations non-Linéaires",

Gauthier-Villars-Paris-(1962).

[3] Roseau, M., "Vibrations Non-linéaires et Théorie de la Stabilité". Springer Tracts, (1966).

[4] Krasnoselsky, M.A., "Functional-Analytic Methods in The Theory of Nonlinear Oscillations", In Proceedings of Fifth Internat. Conf. On Nonlinear Oscillation (Kiev, 1969) 1, Izdanie Inst. Mat. Akzd. Nauk Ukrain SSR, Kiev (1970).

[5] Karmov, L.N., "The Approximate Finding of Periodic Motions of Autonomous Systems, Izvestiya Vuzov Radiofizica 11, No. 7 (1968)

[6] Nayfeh, A.H., 1- "Perturbation Methods", Wiley Interscience, New York (1973). 2- "Introduction to Perturbation Techniques", Wiley Interscience, New York (1981).

[7] Hartwal, H., Malik, A.K., and GHOSH, A., "Forced nonlinear oscillations of an Autoparametric System", Journal of Applied Mechanics, 50, (1983).

[8] Samoilenko, A.M., Ronto, N.I., "Numerical-Analytical Methods of Investigating Periodic Solutions", English translation, Mir Publishers, (1979).

[9] Stokes, A., "On the approximation of Nonlinear Oscillations", In Proceedings of Internat. Symposium on Nonlinear Oscillations, Izdat. Akad. Nauk Ukrain. SSR: Kiev (1963).

[10] Stokes, A., "On the approximation of Nonlinear Oscillations", Journal of Differential Equations 12, No. 3 (1972).

[11] Strygin, V.V., "On the Theory of Auto-Oscillations", Synopsis of a D. Sc. Dissertation, Kiev (1973).

[12] Myshkis, A.D., "Advanced Mathematics for Engineers (Special Courses)". English Translation, Mir Publishers, (1975).

[13] Atherton, D.P., "Stability of Nonlinear Systems", John Wiley, New York, 17-22, (1981).

[14] Cole, J.D., "Pertubation Methods in Applied Mathemtics", Blaisdell, Waltham; (1968).

[15] HAAG, J., 1-"Sur la Synchronisation d'un Système a Plusieurs degrés de Liberté", Annales de l'Ecole Normale Superieur, LXIV 285-338, (1947).
2- "Sur la Synchronisation Harmonique et Sous-Harmonique", Annales Françaises de Chronometrie, 201-264, (1947).
3-"Les mouvements Vibratoires", Tome 1, Press Universitaires de France, (1952).

[16] Chaléat, R., 1-"Recherches sur la Synchronisation", Annales Françaises de Chronometrie, Paris, 167-234 et 261-327, (1954).
2- "La Théorie Génerale de la Synchronisation", Acte de Congrés International de Chronometrie, Tome 1,295-312, Paris (1954).
3- "Troisième Approximation de la Périod d'un Oscillateur Non-Linéaire, Comptes Rendus, t. 256, (1963).

[17] Balbi, J.H., "Approximations Supérieurs de la Théorie de la

Synchronisation de J. Haag", Thèse, Besançon, France, (1971).
[18] Malkin, I.G., "Some Problems in the Theory of Nonlinear Oscillations", Fizmatgiz, Moscow, (1956). The English Translation, United States Atomic Energy Commission, Technical Information Service, ABC-tr-3766, (1959).
[19] Kevorkian, J.; Cole, J.D., "Perturbation Methods in Applied Mathematics", Springer-Verlag, New York, Inc. (1981).
[20] Cunningham, W.J., "Analyse Non-Lineaire, Resolution des Equations Differentiales", Dundon, Paris, (1963).