Alex. Eng. J., Alex. Univ., Volume 28 No.3 , pp. 223-239 (1989)

A DIRECT CUBIC SPLINE APPROACH FOR INITIAL VALUE PROBLEMS

Mohamed Naim Yehia Anwar
Department of Engineering Mathematics and Physics
faculty of engineering, Alexandria University
Alexandria, Egypt

Abstract

In this paper, a direct cubic spline approach for the solution of initial value problems governed by ordinary differential equations is formulated. Error bounds for the function, and its first three derivatives are derived at any value of the independent variable. Numerical results are provided to demonstrate the effectiveness of the method.

1. Intrduction

Numerical methods for ordinary differential equations are in general divided into discrete and continuous ones. Among such continuous methods are those that use splines. Such methods are considered as well established approach for solving ordinary differential equations (see for example [3], [4], and [5]).

In the present work, a direct cubic spline approach for approximating the solution of initial value problems governed by ordinary differential equations is formulated. The general outline of the resulting algorithm is similar to that given in [3]; nevertheless the construction of the spline approximation is different and furthermore error bounds for the function solution and its first three derivatives are derived.

2. The Considered Direct Cubic Spline

Let x_i, $i=0,1,\ldots, N$ be a uniform partition of the interval $[a,b]$. Denote by $S^{(2)}_{N,3}$ the linear space of cubic splines $s(x)$ such that

$$s(x) \in C^2[a,b] \quad ,$$

and $s(x)$ is a cubic polynomial in each subinterval $[x_i, x_{i+1}]$. Set $h = x_{i+1} - x_i$ $(i=0,1,\ldots, N-1)$. If $\varphi(x)$ is a real valued function defined in $[a,b]$ then φ_i, stands for

$$\varphi(x_i) , \quad (i = 0 ,1,\ldots , N)$$

Theorem 1 :

There exists a unique cubic spline $s(x) \in S^{(2)}_{N,3}$ whose first derivatives s'_i ,$(i=0,1,\ldots,N)$, are given along with s_o and s''_o. This spline is defined in $[x_i, x_{i+1}]$ by

$$s(x) = s_i\, W_o(t) + hs'_i\, W_1(t) + hs''_i\, W_2(t) + hs'_{i+1} W_3(t) \quad , \tag{2.1}$$

where

$$W_o(t) = 1 \ , \ W_1(t) = t - \frac{t^3}{3}, \ W_2(t) = \frac{t^2}{2} - \frac{t^3}{3} \ , \ W_3(t) = \frac{t^3}{3} \ , \tag{2.2}$$

and $t = (x - x_i)/h$.

The coefficients s_i, s''_i in (2.1) are given by the recurrence formulae $(i=1,2,\ldots,N)$

$$s''_i = - s''_{i-1} + \frac{2}{h} (- s'_{i-1} + s'_i) \quad ,$$

$$s_i = s_{i-1} + \frac{h}{3} (2s'_{i-1} + s'_i) + \frac{h^2}{6} s''_{i-1} \quad , \tag{2.3}$$

where: s_o, s''_o are known.

Proof

If $P_3(t)$ is a cubic polynomial in $[0,1]$ then it can be expressed as

$$P_3(t) = P_3(0)\, W_o(t) + P'_3(0)\, W_1(t) + P''_3(0)\, W_2(t) + P'_3(1)\, W_3(t) \ .$$

Using $P_3(t)=1,t,t^2$ and t^3, the corresponding expressions in (2.2) for W_o, W_1, W_2 and W_3 can be obtained from the resulting linear system of equations.

Now for a fixed $i \in \{0,1 \dots, N-1\}$, set $x=x_i+th$, $0 \leq t \leq 1$ and thus we get the expression of $s(x)$ given in (2.1). We have a similar expression for $s(x)$ in $[x_{i-1}, x_i]$. Since $s(x) \in C[a,b]$, so the continuity conditions $s(x_i^-)=s(x_i^+)$ and $s''(x_i^-)=s''(x_i^+)$ lead to the above recurrence formulae in (2.3), above. This completes the proof.

3. Algorithm For Initial Value Problem

Consider the initial value problem,

$$y' = g(x,y), \; a \leq x \leq b, \; y(a) \text{ is given} \qquad (3.1)$$

We suppose that g is defined and continuous in $[a,b] \times R$ and it has a continuous bounded derivative, with respect to y. That is,

$$|g_y(x,y)| \leq L \text{ in } [a,b] \times R \qquad (3.2)$$

This guarantees the existence and uniqueness of the solution denoted by $y = f(x)$ in (3.1).

We want to approximate $y=f(x)$ by the cubic spline defined by (2.1) to (2.3) of section 2.

The solution $y=f(x)$ of the initial value problem (3.1) is approxiamted in $[a,b]$ by the cubic spline $s(x) \in S^{(2)}_{N,3}$ defined by the following algorithm;

1) Compute s'_o, s''_o from

$$s'_o = y'(a) = f'_o = g(a,f_o) = g(a,s_o) ,$$

$$s''_o = y''(a) = f''_o = g_x(a,s_o) + g_y(a,s_o) . s'_o , \tag{3.3}$$

where $s_o = y(a) = f_o$.

2) For $i = 1,2,\ldots,N$ compute

$$s'_i = g(x_i,s_i) ,$$

$$s_i = s_{i-1} + \frac{h}{3}(2 s'_{i-1} + s'_i) + \frac{h^2}{6} s''_{i-1} , \tag{3.4}$$

$$s''_i = -s''_{i-1} + \frac{2}{h}(- s'_{i-1} + s'_i) ,$$

This is equivalent to

$$s_i = s_{i-1} + \frac{h}{3}[2g(x_{i-1}, s_{i-1}) + g(x_i,s_i)] + \frac{h^2}{6} s''_{i-1} ,$$

$$s''_i = - s''_{i-1} + 2[-g(x_{i-1},s_{i-1}) + g(x_i, s_i)]/h , \tag{3.5}$$

3) In each subinterval $[x_i,x_{i+1}]$, $i = 0,1,\ldots, N-1$ let

$$s(x) = s_i W_o(t) + hs'_i W_1(t) + hs''_i W_2(t) + hs'_{i+1} W_3(t), \tag{3.6}$$

where $t = (x-x_i)/h$, W_0,W_1,W_2, W_3 are given by (2.2) .

Notice that if g is nonlinear with respect to its arguments x and y then (3. 5) is nonlinear in s_i. The solution is obtained by the fixed point iteration

$$s_i^{(p+1)} = s_{i-1} + h[2g(x_{i-1}, s_{i-1}) + g(x_i, s_i^{(p)})]/3 + h^2 s''_{i-1}/6 . \quad (3.7)$$

It will be shown (see section 4) that this iteration converges for any $s_i^{(o)}$ to the same limit provided that $h < 3/L$.

4. Convergence of The Algorithm and Error Bounds

In this section we give L_∞ bounds for $(s(x) - f(x))$ and its first three derivatives in $[a,b]$. The following analysis is carried out for the general nonlinear case. $\|\cdot\|$ denotes the L_∞ norm, $e^{(r)}(x)$ is $s^{(r)}(x) - f^{(r)}(x)$, $r = 0, 1,2,3$ where r denotes the order of differentiation.

Lemma 1 Let s_i and s'_i $(i = 0, 1, \ldots, N)$ be given by (3.4) then,

$$|e'_i| \leq L \; |e_i| \quad . \qquad (4.1)$$

Proof: We have from (3.1) and (3.4)

$$s'_i = g\,(x_i, s_i), \; f'_i = g\,(x_i, f_i) \quad .$$

Thus by the mean value theorem

$$s'_i - f'_i = g_y(x_i, \delta_i)(s_i - f_i) \quad ,$$

where δ_i is between s_i and f_i. Hence (4.1) follows from the condition (3.2).

Lemma 2: If $h < 3/L$ then ,

$$|e_1| \leq 9h^4 \left\| f^{(4)} \right\| / [64(3-hL)] \quad , \qquad (4.2)$$

provided $f \in C^4[a,b]$.

Proof: Equation (3.4) can be rearranged in the following form

$$s_1 = [s_o + h(2f'_o + f'_1)/3 + h^2 s''_o/6] + h(2e'_o + e'_1)/3 \quad .$$

Since $s_o = f_o$ and $s''_o = f''_o$ from (3.3), thus by subtracting f_1 from both sides of the above equality and expanding the right hand side about $x_o + h/2$ using Taylor's expansion of order 4 we get

$$|e_1| \leq 3h^4 \left\| f^{(4)} \right\| /64 + h(2\,|e'_o| + |e'_1|)/3 \quad .$$

Using (4.1) and $e'_o = 0$ we obtain (4.2) .

Theorem 2 Let s_i $(i = 0, 1, \ldots, N)$ be given by (3.4) and for $h < 3/L$ and $f \in C^5[a,b]$ we have the following error bound $(i = 1, 2, \ldots, N)$

$$|e_i| \leq [9h^4 \left\| f^{(4)} \right\| /(64(3-hL)] \, \mathrm{Exp}\, [6L(x_i - a)/(3-hL)] +$$

$$h^4 \left\| f^{(5)} \right\| /[180L \{ \mathrm{Exp}\, (6L(x_i - a)/(3-hL)) - 1 \}] . \qquad (4.3.1)$$

Furthermore if $h < 2/L$ then (4.3.1) leads to ,

$$|e_1| \leq [(9 \left\| f^{(4)} \right\| /64) + \left\| f^{(5)} \right\| /(180L)] \, h^4 \, \mathrm{Exp}\, [6L(b-a)]$$

$$h^4 \left\| f^{(5)} \right\| /(180L) \qquad (4.3.2)$$

Proof: From (3.4) we have (i = 1,2, ..., N-1)

$$s_i = s_{i-1} + h(2s'_{i-1}+s'_i)/3+h^2 s''_{i-1}/6 \quad ,$$

$$s_{i+1} = s_i + h(2s'_i+s'_{i+1})/3+h^2 s''_i/6 \quad .$$

Adding these two equations and using the third equation of (3.4)we get (i = 1,2,..., N-1)

$$s_{i+1} = s_{i-1} + h\,(s'_{i-1} + 4s'_i + s'_{i+1})/3 \quad , \qquad (4.4)$$

which can be rewritten as

$$s_{i+1} = e_{i-1} + f_{i-1} + h\,(f'_{i-1}+4f'_i+f'_{i+1})/3+h(e'_{i-1}+4e'_i+e'_{i+1})/3 \quad .$$

Subtracting f_{i+1} from both sides of the above equation and noticing that ,

$$f_{i-1}-f_{i+1}+h(f'_{i-1}+4f'_i+f'_{i+1})/3= \int_{x_{i-1}}^{x_{i+1}} f'(u)du+h(f'_{i-1}+4f'_i+f'_{i+1})/3 \quad ,$$

which is the error term of the classical Simpson's rule over two subintervals, we obtain ,

$$|e_{i+1}| \le |e_{i-1}| + h^5 \|f^{(5)}\|/90 + h(\,|e'_{i-1}| +4|e'_i|+|e'_{i+1}|\,)/3 \quad . \qquad (4.5)$$

Using (4.1) and since h < 3/L we get ,

$$|e_{i+1}| \le \alpha\, |e_i| + \beta\, |e_{i-1}| + \sigma \qquad (4.6)$$

where

$\alpha = 4hL/(3-hL)$, $\beta = (3+hL)/(3-hL)$, and $\sigma = h^5 \| f^{(5)} \| [30(3-hL)]$. (4.7)

Then (4.6) is equivalent to ,

$$E_{i+1} \leq AE_i + \sigma V \qquad (4.8)$$

where

$$E_i = \begin{bmatrix} |e_i| \\ |e_{i-1}| \end{bmatrix}, \quad i = 1,2,\ldots, N ,$$

$$A = \begin{bmatrix} \alpha & \beta \\ 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

From (4.8) we can arrive at ,

$$E_{i+1} \leq \| A \|^{i} \| E_1 \| + \sigma (\| A \|^{i} - 1) . \qquad (4.9)$$

But

$$\| A \| = \alpha + \beta = (3+5hL)/(3-hL) = 1 + 6hL/(3-hL) .$$

Thus

$$\| A \|^{i} = [1 + 6hL/(3-hL)]^{i} \leq \text{Exp} [6ihL/(3-hL)] . \qquad (4.10)$$

But $\| E_1 \| = \max (|e_o|, |e_1|) = |e_1|$, since $e_o = 0$. Hence (4.9) and (4.10) give ,

$$\| e_i \| \leq |e_1| \text{Exp}[6L(x_i - a)/(3-hL)] +$$

$$[\sigma (3-hL)/(6hL)]\{ \text{Exp} [6L(x_i - a)/(3-hL)] - 1 \} .$$

Using (4.2) and (4.7) we arrive at the error bound (4.3.1). When $hL < 2$ that is $3-hL > 1$ and since $x_i - a \leq b-a$ then (4.3.2) follows. This completes the proof.

Notice that an error bound on e' can be obtained by multiplying that for e by L (Lemma 1).

Lemma 3: If $h < 3/L$ then

$$\left| e''_1 \right| \leq \left[\frac{1}{6} + \frac{9hL}{32(3-hL)} \right] h^2 \left\| f^{(4)} \right\| \tag{4.11}$$

provided $f \varepsilon C^4 [a,b]$

Proof: We have, $s'_o = f'_o$, $s''_o = f''_o$. For $i = 1$, (3.4) gives,

$$s''_1 - f''_1 = - f''_o - f''_1 + \frac{2}{h} (-f'_o + f'_1) + \frac{2}{h}(-e'_o + e'_1) ,$$

or

$$e''_1 = -f''_o - f''_1 + \frac{2}{h} \int_a^{a+h} f''(t)dt + \frac{2}{h} (-e'_o + e'_1) .$$

Integrating by parts twice we get

$$e''_1 = \int_a^{a+h} \left[- \frac{h}{4} + \frac{1}{h} (t - \frac{h}{2})^2 \right] f^{(4)} (t)\, dt + \frac{h}{2} (-e'_o + e'_1) .$$

Now by the mean value theorem for integrals, (4.1) and (4.2) we obtain

$$\left| e''_1 \right| \leq h^2 \left\| f^{(4} \right\| / 6 + 9h^3 L \left\| f^{(4)} \right\| / [32(3-hL)] ,$$

which is (4.11).

Theorem 3: Let s''_i (i=0,1,..., N) be given by (3.4) and for $h < 3/L$ and $f \in C^5[a,b]$ we have the following error bound, (i=1,2,...,N),

$$|e''_i| \leq |e''_1| + (x_i - a)[17h^2 \|f^{(5)}\|/12 + 4L\|e\|/h^2] , \qquad (4.12)$$

where $|e''_1|$ satisfies (4.11), $\|e\| = \max |e_i|$ and , $|e_i|$ satisfies (4.3.1).

Proof: From (3.4) we can write (i=1,2,..., N-1)

$$s''_i = 2(-s'_{i-1} + s'_i)/h - s''_{i-1} ,$$

and

$$s''_{i+1} = 2(-s'_i + s'_{i+1})/h - s''_i .$$

Subtracting these two equations and rearranging, we arrive at,

$$e''_{i+1} = e''_{i-1} + f''_{i-1} - f''_{i+1} + 2\ (f'_{i-1} - 2f'_i + f'_{i+1})/h + 2(e'_{i-1} - 2e'_i + e'_{i+1})/h .$$

Expanding about x_{i-1} using Taylor's expansion of order 5, we can show that (i=1,2,..., N-1),

$$|e''_{i+1}| \leq |e''_{i-1}| + 17h^3 \|f^{(5)}\|/6 + 8L\|e\|/h , \qquad (4.13)$$

which is used to obtain (4.12).

From (3.6) we have for x restricted in $[x_i, x_{i+1}]$, (i=0,1,...,N-1)

$$s'''(x) = -2s'_i/h^2 - 2s''_i/h + 2s'_{i+1}/h^2 \qquad (4.14)$$

which is constant over this subinterval.

Lemma 4: For i = 0, 1,2,.., N we have,

$$|e'''_i| \leq 2h\|f^{(4)}\|/3+17h\|f^{(5)}\|/6+4L\|e\|h^2+2\|e''\|/h \quad , \qquad (4.15)$$

provided that $f \in C^5[a,b]$ and s'''_i, is defined as in (4.14).

Proof: From (4.4) we can write,

$$e'''_i = -f'''_i-2f'_i/h^2-2f''_i/h+2f'_{i+1}/h^2 - 2e'_i/h^2-2e''_i/h+2e'_{i+1}/h^2 \ .$$

Expanding the right hand side using Taylor's expansion of order 5 about x_i, we get the result (4.15).

Theorem 4: Let s(x) be the cubic spline defined in (3.6). If $h < 3/L$ and $f \in C^5[a,b]$ then for any $x \in [a,b]$ we have,

$$|e^{(r)}(x)| \leq \begin{cases} (1+hL)\|e\|+h^2\|e''\|/6+7h^4\|f^{(4)}\|/72 & r = 0 \\ 5L\|e\|+h\|e''\|+h^3\|f^{(4)}\|/6 & r = 1 \\ 4L\|e\|/h+\|e''\|+h^2\|f^{(4)}\|/6 & r = 2 \\ 4L\|e\|/h^2+2\|e''\|/h+4h\|f^{(4)}\|/3 \ , & r = 3 \end{cases} \qquad (4.16)$$

where $e^{(r)}(x) = s^{(r)}(x) - f^{(r)}(x)$, $\|e\|$ and $\|e''\|$ are bounded by (4.3.1) and (4.12) respectively.

Proof: Subtracting f(x) from both sides of (3.6) and expanding the right hand side about x_i using Taylor's expansion of order 4 we get,

$$e(x) = e_i + h^2(t^2/2 - t^3/3)e''_i + h[(t - t^3/3)e'_i + t^3 e'_{i+1}/3] +$$

$$+ h^4 [t^3 f^{(4)}(\mu_i)/18 - t^4 f^{(4)}(\delta_i)/24] \quad .$$

Using (4.1) and since $0 \le t \le 1$ we can establish the first result in (4.16).

From (3.6) we obtain,

$$s''(x) = s'_i(-2t)/h + s''_i(1-2t) + s'_{i+1}(2t)/h \quad .$$

Subtracting f"(x) from both sides and expanding about x_i as above we can arrive at the third result in (4.16),(r=2).

Similarly the fourth result that corresponds to r=3, follows from (4.14) by substracting f"' (x) from both sides and expanding as above.

Finally, we can write,

$$s'(x) - f'(x) = \int_{x_i}^{x} [s''(\theta) - f''(\theta)]d\theta + (s'_i - f'_i) \quad ,$$

from which we have ,

$$|e'(x)| \le \int_{x_i}^{x} |e''(\theta)| \, d\theta + L \, \| e \| \quad .$$

Using (4.16), r=2, we can prove the second result in (4.16),(r=1). Which completes the proof.

5. Convergence of The Associated Fixed Point Iteration

Consider the fixed point iterative process (3.7) which, on writing $v_p = s_i^{(p)}$, (i is fixed) may be expressed as,

$$v_{p+1} = \varphi(v_p) \quad , \qquad (5.1)$$

where φ maps R into R and φ is contracting for $h < 3/L$ since from (3.2) and (3.7) it can be shown that,

$$\left|\varphi(u) - \varphi(v)\right| = h\left|g(x_i, u) - g(x_i, v)\right|/3 \leq hL \left|u-v\right|/3 \quad .$$

Note that $h < 3/L$ is the same condition as was necessary for the validity of all the previous error bounds. It follows then from the classical fixed point problem and associated iteration that (5.1) converges to its unique fixed point (which is s_i) for any initial guess $s^{(o)}_i$. A suitable choice would be s_{i-1}.

6. Numerical Results

In order to demonstrate the performance of the proposed method the following numerical examples are considered:

Example 1

$$y' = 1/x^2 - y/x - y^2, \quad 1 \leq x \leq 2 \ ,$$

and $y(1) = -1$. The exact solution is $y=f(x) = -1/x$.

Table 1 gives the maximum error bounds for $\left|s^{(r)}_i - f^{(r)}(x_i)\right|$, r=0, 1,2, and 3 for different choices of the step size h. For each values of r, the entries in the top row are calculated values while those in the second one are the estimated ones based on the theoretical error bounds.

Table 1

Maximum error bounds for $\|\|s^{(r)}-f^{(r)}\|\|$	h=.1	h=.05	h=.025	h=.01
r=0	2.9×10^{-5}	2.4×10^{-6}	2.4×10^{-7}	3.0×10^{-7}
	1.2×10^{-1}	4.9×10^{-3}	2.5×10^{-4}	5.8×10^{-6}
r=1	2.6×10^{-5}	2.0×10^{-6}	1.8×10^{-7}	2.4×10^{-7}
	7.5×10^{-3}	9.8×10^{-4}	1.2×10^{-4}	1.2×10^{-5}
r=2	3.1×10^{-2}	8.8×10^{-3}	2.3×10^{-3}	4.0×10^{-4}
	$.18\times10$	4.5×10^{-1}	1.1×10^{-1}	5.3×10^{-2}
r=3	$.12\times10$	6.9×10^{-1}	3.7×10^{-1}	1.6×10^{-1}
	3.6×10	1.8×10	$.9\times10$	$.37\times10$

Example 2

$$y' = -x\,y^2 \quad , 2 \le x \le 3 \quad ,$$

and $y(2) = 1$. The exact solution is $y = f(x) = 2/(x^2-2)$.

Table 2 gives the maximum error bounds for $\left|s^{(r)}_i - f^{(r)}(x_i)\right|$, $r = 0, 1, 2$, and 3 as before.

Table 2

Maximum error bounds for $\lVert s^{(r)}-f^{(r)} \rVert$	h=.1	h=.05	h=.025	h=.01
r=0	3.6×10^{-4}	2.4×10^{-5}	5.2×10^{-6}	7.2×10^{-7}
	5.3×10^{3}	5.8×10	$.2\times10$	3.0×10^{-2}
r=1	7.8×10^{-4}	6.3×10^{-5}	1.6×10^{-5}	2.0×10^{-6}
	8.3×10^{-2}	1.0×10^{-2}	1.3×10^{-3}	9.4×10^{-5}
r=2	4.2×10^{-1}	1.1×10^{-1}	2.8×10^{-2}	4.3×10^{-3}
	8.7×10	2.2×10	$.6\times10$	$.1\times10$
r=3	1.7×10	1.5×10	1.4×10	1.2×10
	1.7×10^{3}	8.7×10^{2}	4.3×10^{2}	1.7×10^{2}

Conclusion

The above results show that calculated error bounds are within the theoretical bounds obtained earlier. Several other numerical tests have been performed and results are found to be in agreement with the theoretical analysis presented in the previous sections. However due to the length limitation only the above tables are reported here.

References

[1] Anwar M.N., El Tarazi M.N. " Direct cubic spline with application to quadrature". Submitted for publication.

[2] Henrici P. "Discrete variable methods in ordinary differential equations". John Wiley & Sons, Inc. 1962.

[3] Lascalzo F.R., Talbot T.D. "Spline function approximations for solutions of ordinary differential equations" SIAM. J. Numer. Anal. Vol. 4, No.3, 1967.

[4] Werner H. "Interpolation and integration of initial value problems of ordinary differential equations by regular splines". SIAM. J. Numer. Anal. Vol. 12, No.2, 1975.

[5] Werner H. "Spline functions and the numerical solutions of differential equations". Special Topics of applied mathematics. Proceedings of the seminar held at the GMD, Bonn, 8-10 October, 1979. Eds. Frehse J., Pallaschke D., and Trottenberg U. North Holland NY. 1980.