Alex. Eng. J., Alex. Univ., Volum, 28 No. 2, PP. 171 - 189 (1989)

SIMILARITY ANALYSIS OF A SUBMERGED PLANE TURBULENT JET DISCHARGED OBLIQUELTY TO THE SURROUNDINGS

A.A. Abdel-Rahman

Assistant Professor of Mechanical Engineering

Faculty of Engineering, Alexandria University

Alexandria, Egypt

Abstract

Similarity analysis of a buoyant turblent plane jet discharged obliquely to the surroundings is undertaken by the present paper. The analysis is performed using the equations of the mean motion in the curvilinear coordinate system. The similarity conditions for the three jet regions are deduced and their implications are discussed. The results of the analysis indicate that the jet flow is self-similar in both the near and far regions. The analysis further shows that the intermediate region of the jet does not exhibit self-similarity.

Introduction

Similarity analysis has been used and shown to be a helpful tool in giving information on the general behaviour of both isothermal and heated jets [1,2]. Self-similarity, when it exists in a flow field, can tell how the data will likely behave and also, provides an unified way for presenting experimental data from different experiments. For isothermal and vertical heated (buoyant) jets, similarity analysis and similarity conditions have been derived and well established.

To data, no attempt has been made towards the similarity analysis for the tilted buoyant jet. The present paper, therefore, concerns itself with the similarity analysis of the mean flow of a buoyant turbulent plane jet discharged to the surroundings other than vertically upwards. The trajectory of such a jet flow will continue to deviate from its initial direction due to the driving action of the buoyancy force, as shown in Fig. (1).

The jet flow in such a case may be divided into three flow regions, based on the relative role of the forces acting on the flow. The three jet regions are: the near field (non-buoyant) region which is dominated by the inertia forces and where the jet direction coincides with the initial discharge direction and the raduis of curvature of the jet center-line goes to ∞, the intermediate region where the buoyancy forces and inertia forces are of comparable magnitude and the raduis of curvature of the jet center-line is most significant, and the far field (buoyant) region where buoyancy forces dominate and the raduis of curvature of the jet center-line goes to ∞ again.

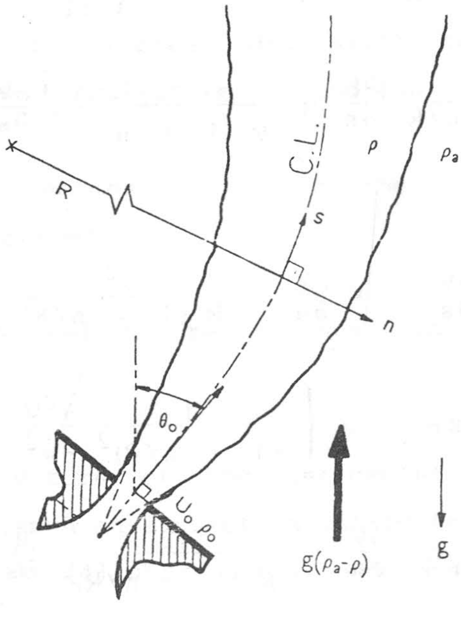

Fig. (1) A curved buoyant plane jet

1.1 Equations of Motion

The basic equations of motion of the flow according to an appropriate coordinate system (curvilinear coordinate system) for a constant property steady plane (2-dimensional) jet are:

$$\frac{\partial U}{\partial s} + \frac{\partial}{\partial n}\left[\left(1 + \frac{n}{R}\right) V\right] = 0 \tag{1}$$

$$\frac{U}{(1+n/R)}\frac{\partial U}{\partial s} + V\frac{\partial U}{\partial n} + \frac{UV}{R(1+n/R)} =$$

$$-\frac{1}{\rho(1+n/R)}\frac{\partial P}{\partial s} + g_s + \nu\left\{\frac{1}{(1+n/R)^2}\frac{\partial^2 U}{\partial s^2} + \frac{\partial^2 U}{\partial n^2}\right. \qquad (2)$$

$$\left. + \frac{1}{R(1+n/R)}\frac{\partial U}{\partial n} + \frac{2}{R(1+n/R)^2}\frac{\partial V}{\partial s} - \frac{U}{R^2(1+n/R)^2}\right\}$$

$$\frac{U}{(1+n/R)}\frac{\partial V}{\partial s} + V\frac{\partial V}{\partial n} - \frac{U^2}{R(1+n/R)} =$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial n} + g_n + \nu\left\{\frac{1}{(1+n/R)^2}\frac{\partial^2 V}{\partial s^2} + \frac{\partial^2 V}{\partial n^2}\right. \qquad (3)$$

$$\left. + \frac{1}{R(1+n/R)}\frac{\partial V}{\partial n} - \frac{2}{R(1+n/R)^2}\frac{\partial U}{\partial s} - \frac{V}{R^2(1+n/R)^2}\right\}$$

$$\frac{\partial UT}{\partial s} + \left(1+\frac{n}{R}\right)\frac{\partial VT}{\partial n} + \frac{VT}{R} = \frac{k}{\rho c_p}\left\{\frac{1}{(1+n/R)^2}\frac{\partial^2 T}{\partial s^2} + \frac{\partial^2 T}{\partial n^2}\right\} \qquad (4)$$

These are the instantaneous equations of continuity, momentum in s and n directions and thermal energy, respectively Refer to [3] for a detailed derivation of these equations. The term g_s ($= -g\cos\theta$)is the component of the gravity force in the s-direction and g_n ($=g\sin\theta$) is the component in n-direction.

To obtain the 2-dimensional mean flow equations, Reynolds decomposition is applied on U,V, P and T (e.g. U = U + u, etc.) in the equations of motion, and then time-averaging. Some assumptions are made in the process of obtaining these time-averaged equations, they are:

- The viscous terms and thermal diffusivity terms are negligibly small compared to the turbulent terms, and can be dropped out .

- Neglecting density changes except in the buoyancy term (the "Boussinesq approximation").

- The density is assumed to be a linear function of temperature (perfect gas).

- The raduis of curvature of the jet center-line, R, is assumed constant. However, such an assumption introduces stight changes to the viscous terms only [4] which are already dropped out.

With the above-mentioned assumptions in mind, the time-aneraged forms of equations (1) to (4) are:

$$\frac{\partial \bar{U}}{\partial s} + \frac{\partial}{\partial n}\left[\left(1 + \frac{n}{R}\right) \bar{V}\right] = 0 \tag{5}$$

$$\bar{U}\frac{\partial \bar{U}}{\partial s} + \left(1 + \frac{n}{R}\right)\bar{V}\frac{\partial \bar{U}}{\partial n} + \frac{\bar{U}\bar{V}}{R} = g\,\beta\cos\theta\left(1 + \frac{n}{R}\right)(\bar{T} - T_a)$$

$$- \frac{1}{\rho}\frac{\partial \bar{P}}{\partial s} - \frac{\partial \overline{u^2}}{\partial s} - \left(1 + \frac{n}{R}\right)\frac{\partial \overline{uv}}{\partial n} - 2\,\frac{\overline{uv}}{R} \tag{6}$$

$$\bar{U}\frac{\partial \bar{V}}{\partial s} + \left(1 + \frac{n}{R}\right)\bar{V}\frac{\partial \bar{V}}{\partial n} - \frac{\bar{U}^2}{R} = -g\,\beta\sin\theta\left(1 + \frac{n}{R}\right)(\bar{T} - T_a)$$

$$- \left(1 + \frac{n}{R}\right)\frac{1}{\rho}\frac{\partial \bar{P}}{\partial n} - \frac{\partial \overline{uv}}{\partial s} - \left(1 + \frac{n}{R}\right)\bar{V}\frac{\partial \overline{v^2}}{\partial n} + \frac{\overline{v^2} - \overline{u^2}}{R} \tag{7}$$

$$\frac{\partial \bar{U}\bar{T}}{\partial s} + \left(1 + \frac{n}{R}\right)\frac{\partial \bar{V}\bar{T}}{\partial n} + \frac{\bar{V}\bar{T}}{R} = \frac{\partial \overline{ut}}{\partial s} - \left(1 + \frac{n}{R}\right)\frac{\partial \overline{vt}}{\partial n} - \frac{\overline{vt}}{R} \tag{8}$$

in these equations is the coefficient of volumetric (thermal expansion, $\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \equiv \frac{1}{T}$ for perfect gases. P is the time-average pressure relative to the ambient pressure.

1.2 Self-similarity analysis

In this section, the self-similar behaviour of the jet flow is examined through analizing the mean flow equations, given in the previous section. The integration of the n-momentum equation, equation (7), with respect to n from n to a point located outside the jet flow yields:

$$\int_{n}^{\infty} \frac{\bar{U}^2}{R\left(1 + \frac{n}{R}\right)}\, dn = -\frac{\bar{P}}{\rho} + g\,\beta\,\sin\theta \int_{n}^{\infty} (\bar{T} - T_a)\, dn - \overline{v^2} \qquad (9)$$

Differentiating the above equation with respect to s, we get

$$-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial s} = \frac{\partial}{\partial s}\left[\int_{n}^{\infty} \frac{\bar{U}^2}{R\left(1 + \frac{n}{R}\right)}\, dn\right] - g\beta \frac{\partial}{\partial s}\left[\sin\theta \int_{n}^{\infty} (\bar{T} - T_a)\, dn\right] + \frac{\partial \overline{v^2}}{\partial s} \qquad (10)$$

Then, substituting equation (10) into the s-momentum equation, equation (6), gives:

$$\bar{U}\frac{\partial \bar{U}}{\partial s} + \left(1 + \frac{n}{R}\right)\bar{V}\frac{\partial \bar{U}}{\partial n} + \frac{\bar{U}\bar{V}}{R} = \frac{\partial}{\partial s}\int_{n}^{\infty} \frac{\bar{U}^2}{R\left(1 + \frac{n}{R}\right)}\, dn + g\,\beta\,\cos\theta\left(1 + \frac{n}{R}\right)(\bar{T} - T_a) - \left(1 + \frac{n}{R}\right)\frac{\partial \overline{uv}}{\partial n} - g\,\beta\,\frac{\partial}{\partial s}\left[\sin\theta \int_{n}^{\infty} (\bar{T} - T_a)\, dn\right] - \frac{\partial\left(\overline{u^2} - \overline{v^2}\right)}{\partial s} \qquad (11)$$

Now, the jet flow is considered to be self-similar if one velocity, one temperature and one length scale make the time-averaged quantities in the equations of motion dimensionless function of one geometrical variable only. In other words, self-similarity will exist if:

$$\bar{U} = \bar{U}_c \; f_1 \; (\eta) \; , \tag{12}$$

$$\overline{u^2} = \bar{U}_c^2 \; f_2 \; (\eta) \; , \tag{13}$$

$$\overline{v^2} = \bar{U}_c^2 \; f_3 \; (\eta) \; , \tag{14}$$

$$\overline{vu} = \bar{U}_c^2 \; f_4 \; (\eta) \; , \tag{15}$$

$$\Delta\bar{T} = \Delta\bar{T}_c \; f_5 \; (\eta) \; , \tag{16}$$

$$\overline{ut} = \bar{U}_c \; \Delta\bar{T}_c \; f_6 \; (\eta) \tag{17}$$

$$\overline{vt} = \bar{U}_c \; \Delta\bar{T}_c \; f_7 \; (\eta) \tag{18}$$

$$\eta = \frac{n}{b} \tag{19}$$

Following the work which has been done in the vertical jets and plumes, b is considered the value of n at which $\bar{U} = 0.5\ \bar{U}_c$.
The mean velocity distribution in the present curved jet flow, unlike the vertical case, is unsymmetrical and, therefore, the value of n at which $U = 0.5\ U_c$ on either side of the jet center-line is different. However, it is assumed that the value of b on both sides of the jet is nearly the same; and assumption which has been reasonably justified experimentally [3].

To find an expression for the transverse mean velocity,V, the continuity equation, equation (5), is integrated to give:

$$\left(1 + \frac{n}{R}\right) \bar{V} = - \int_0^n \frac{\partial \bar{U}}{\partial s}\, dn \qquad (20)$$

Making use of equations (12) to (19), equation (20) becomes:

$$\left(1 + \frac{n}{R}\right) \bar{V} = \bar{U}_c \frac{db}{ds} \left(\eta f_1 - \int_0^n f_1 d\eta\right) - \frac{d\bar{U}_c}{ds}\, b \int_0^n f_1\, d\eta \qquad (21)$$

Substituting equation (21) into equations (11) and (8), transforming them into dimensionless form and multiplying the first by $b/\bar{U}_c^2\ (1+ b/R)$ and the second by $b/\bar{U}_c \Delta \bar{T}_c$, yields.

$$f_4' = \left\{\frac{\frac{d\bar{U}_c}{ds}\ b}{\bar{U}_c\ \left(1+\frac{\eta b}{R}\right)}\right\}\left[f_1'\int_0^\eta f_1 d\eta\ -\ f_1^2\ 2\ f_3\ -\ 2\ f_2\right]$$

$$+\left\{\frac{\frac{db}{ds}}{\left(1\ +\ \frac{\eta b}{R}\right)}\right\}\left[f_1'\int_0^\eta f_1 d\eta\ +\ \eta\ f_2'\ -\ \eta\ f_3'\right]$$

$$+\left\{\frac{\frac{d\bar{U}_c}{ds}\ b^2}{R\ \bar{U}_c\ \left(1\ +\ \frac{\eta b}{R}\right)^2}\right\}\left[f_1\int_0^\eta f_1 d\eta\right]-\left\{\frac{\frac{db}{ds}\ b}{R\ \left(1\ +\ \frac{\eta b}{R}\right)^2}\right\}$$

$$\left[\eta f_1^2 -\ f_1\int_0^\eta f_1 d\eta\right]+\left\{\frac{g\ \beta\ b\ \Delta\bar{T}_c\ \cos\theta}{\bar{U}_c^2}\right\}\ [f_5]$$

$$+\left(\frac{b}{\bar{U}_c^2\ \left(1\ +\ \frac{\eta b}{R}\right)}\right)\frac{\partial}{\partial s}\left(\frac{b\bar{U}_c^2}{R}\int_\eta^\infty \frac{f_1^2}{\left(1\ +\ \frac{\eta b}{R}\right)}\ d\eta\right)$$

$$-\left(\frac{b}{\bar{U}_c^2\ \left(1\ +\ \frac{\eta b}{R}\right)}\right)\frac{\partial}{\partial s}\left(g\ \beta\ b\ \Delta\bar{T}_c\ \sin\theta\int_\eta^\infty f_5\ d\eta\right) \qquad (22)$$

$$f'_7 = -\left\{\frac{\frac{d\Delta\bar{T}_c}{ds}\, b}{\Delta\bar{T}_c\left(1+\frac{\eta b}{R}\right)}\right\}\left[f_6 + f_1 f_5\right] + \left\{\frac{\frac{db}{ds}}{\left(1+\frac{\eta b}{R}\right)}\right\}\left[f'_5\int_0^{\eta} f_1\, d\eta + \eta\, f'_6\right] + \left\{\frac{\frac{d\bar{U}_c}{ds}\, b}{\bar{U}_c\left(1+\frac{\eta b}{R}\right)}\right\}\left[f'_5\int_0^{\eta} f_1\, d\eta - f_6\right] - \left\{\frac{b}{R\left(1+\frac{\eta b}{R}\right)}\right\}\left[f_7\right] \quad (23)$$

where primes denote differentiation w.r.t. η

The left-hand sides of equations (22) and (23) are functions of only. The right-hand sides must, therefore, be functions of only. Since the terms inside the square brackets are functions of only, the curly-braket terms must also be function of or, at least, independent of s, i.e.,

$$\frac{\frac{db}{ds}}{\left(1+\frac{\eta b}{R}\right)} \propto s^0 \quad (24)$$

$$\frac{b \dfrac{db}{ds}}{R \left(1 + \dfrac{\eta b}{R}\right)^2} \propto s^0 \tag{25}$$

$$\frac{b \dfrac{d\bar{U}_c}{ds}}{\bar{U}_c \left(1 + \dfrac{\eta b}{R}\right)} \propto s^0 \tag{26}$$

$$\frac{b \dfrac{d\Delta\bar{T}_c}{ds}}{\Delta\bar{T}_c \left(1 + \dfrac{\eta b}{R}\right)} \propto s^0 \tag{27}$$

$$\frac{b \; \Delta\bar{T}_c \; \cos\theta}{\bar{U}_c^2} \propto s^0 \tag{28}$$

$$\left(\frac{b}{\bar{U}_c^2 \left(1 + \dfrac{\eta b}{R}\right)}\right) \frac{\partial}{\partial s} \left(\frac{b\bar{U}_c^2}{R} \int_\eta^\infty \frac{f_1^2}{\left(1 + \dfrac{\eta b}{R}\right)} \, d\eta\right) \tag{29}$$

$$- \left(\frac{b}{\bar{U}_c^2 \left(1 + \dfrac{\eta b}{R}\right)}\right) \frac{\partial}{\partial s} \left(g \; \beta \; b \; \Delta\bar{T}_c \; \sin\theta \int_\eta^\infty f_5 \; d\eta\right) \propto s^0$$

2. Discussion and Implications

2.1 The near field (Non-bouant) region

In this region, the inertia forces predominate the flow and the temperature acts as a scalar (tacer). The buoyancy forces are, consequently, negligibly small. Here, the direction of the mean flow is nearly straight and conicides with the jet discharge direction; i.e., R and = const. Hence, the self-similarity conditions, equations (24) to (27), reduce to:

$$\frac{db}{ds} \propto s^0 \tag{30}$$

$$\frac{b \dfrac{d\bar{U}_c}{ds}}{\bar{U}_c} \propto s^0 \tag{31}$$

$$\frac{b \dfrac{d\Delta\bar{T}_c}{ds}}{\Delta\bar{T}_c} \propto s^0 \tag{32}$$

Condition (31) does not apply to this, non-buoyant, region of the jet. Equation (11) may also be reduced to

$$\bar{U}\frac{\partial\bar{U}}{\partial s} + \bar{V}\frac{\partial\bar{U}}{\partial n} = -\frac{\partial\overline{uv}}{\partial n} - \frac{\partial\left(\overline{u^2} - \overline{v^2}\right)}{\partial s} \tag{33}$$

Integrating the above equation across the jet and making use of the continuity and Liebnitz's rule gives

$$\frac{d}{ds}\int_0^\infty \left(\overline{U}_c^2 + \overline{u^2} - \overline{v^2}\right) dn = 0 \tag{34}$$

Using the scaling transformation, it becomes

$$\frac{d}{ds}\left[\overline{U}_c^2 \; b \int_0^\infty (f_1^2 + f_2^2 - f_3^2)\, d\eta\right] = 0 \tag{35}$$

since f_1^2 d and $(f_2^2 - f_3^2)$ d are functions

of only, it therefore follows that

$$\frac{d\,(\overline{U}_c^2 \; b)}{ds} = 0 \;, \;\; i.e.; \tag{36}$$

$$\overline{U}_c^2 \; b = constant$$

The energy equation, equation (8), can also be reduced to

$$\frac{\partial \overline{UT}}{\partial s} + \frac{\partial \overline{VT}}{\partial n} = - \frac{\partial \overline{ut}}{\partial s} - \frac{\partial \overline{vt}}{\partial n} \tag{37}$$

Integrating (37) across the jet, we obtain

$$\frac{d}{ds}\int_0^\infty (\overline{U}\;\overline{T} + \overline{ut})\, dn + (\overline{V}\;\overline{T} + \overline{vt})\Big|_0^\infty = 0 \tag{38}$$

Using the scaling transformation, it can be rewritten as

$$\Delta \overline{T}_c = s^{-1} \tag{39}$$

from which it follows that

$$\overline{U}_c \ \Delta \overline{T}_c \ b = \text{constant} \tag{40}$$

Relation (30), then, gives

$$b \propto s \tag{41}$$

Equations (36) and (41) give

$$\overline{U}_c \propto s^{-\frac{1}{2}} \tag{42}$$

Equation (40) to (42) give

$$\Delta \overline{T}_c \propto s^{-\frac{1}{2}}$$

To summarize, the non-buoyant region of the jet flow is self-similar. The length scale (b), velocity scale (U_c) and temperature scale (T_c) vary in the downstream direction as indicated by the expressions (41), (42) and (43) and shown in Figure (2). The variation in the downstream direction, as expected, is identical to that of a vertical buoyant jet [2].

2.2. The far field (Buoyant) region

In this region (the far field region), the buoyancy forces predominate and the jet flow direction will, eventually, be vertically upward; i.e. $\theta \rightarrow 0$ and $R \rightarrow \infty$. The similarity conditions, therefore, reduce to

$$\frac{db}{ds} \propto s^0 \tag{44}$$

$$\frac{b\,\frac{d\bar{U}_c}{ds}}{\bar{U}_c} \propto s^0 \tag{45}$$

$$\frac{b\,\frac{d\Delta\bar{T}_c}{ds}}{\Delta\bar{T}_c} \propto s^0 \tag{46}$$

$$\frac{b\,\Delta\bar{T}_c}{\bar{U}_c^2} \propto s^0 \tag{47}$$

The integration of energy equation for this region of the jet is identical to equation (40);

$$U_c \quad T_c\, b = \text{const.}$$

from (44),

$$b \propto s \tag{48}$$

dividing (47) by (40) yields

$$\frac{1}{\bar{U}_c^3} = \text{constant} \text{ , i.e.;}$$

$$\bar{U}_c = \text{constant} \tag{49}$$

Relation (40) with b S and U_c = const. gives

$$\frac{d}{ds}\left[\bar{U}_C \;\Delta\bar{T}_C \;b\int_0^\infty (f_1 f_5 + f_6)d\eta\right] = 0 \tag{50}$$

The similarity conditions,(44) to (47), are now readily satisfied by the relations (48) to (50) which show that the flow in this region of the jet is self-similar, demonstrating along with Figure (3), how the different scales vary in the downstream direction. It is worth pointing out that the self-similar conditions are identical to those of the buoyant region in a verical buoyant jet. One more point to be mentioned is that the group b T_c/U_c^2 is, in effect expressing Richardson number divided by g . This suggests, along with condition (47), that Richardson number in the buoyant region is constant. Richardson number was however found to be constant in the vertical buoyant jet [5,6].

2.3 The Intermediate Region

This is the region lying between the non-buoyant and buoyant regions of the jet flow. There, the curvature of the jet trajectory is most significant and the buoyancy and inertia forces are comparable.

The flow in this region, changing its self-similar behaviour from the form existing in the non-buoyant region to that of the buoyant region, is unlikely to be self-similar. However, for further justification, the integration of the energy equation, as was done before for the other two regions, leads to equation (40), which by dividing it by the similarity condition (28) gives

$$\frac{\bar{U}_C^3}{\cos\theta} = \text{constant} \tag{51}$$

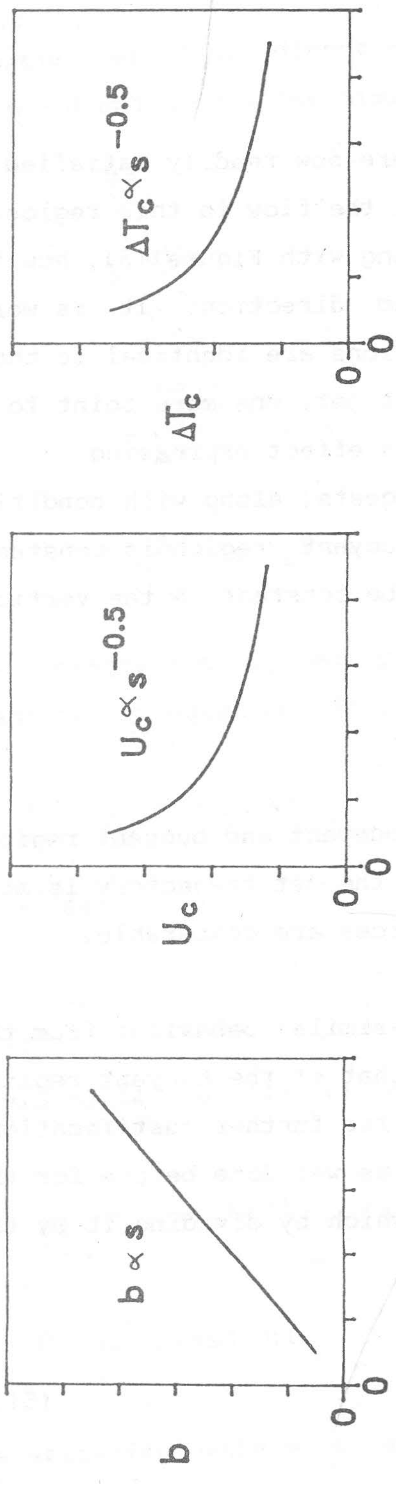
b
0
0
b ∝ s
s
Uc
0
0
Uc ∝ s^-0.5
s
ΔTc
0
0
ΔTc ∝ s^-0.5
s

Fig. 2

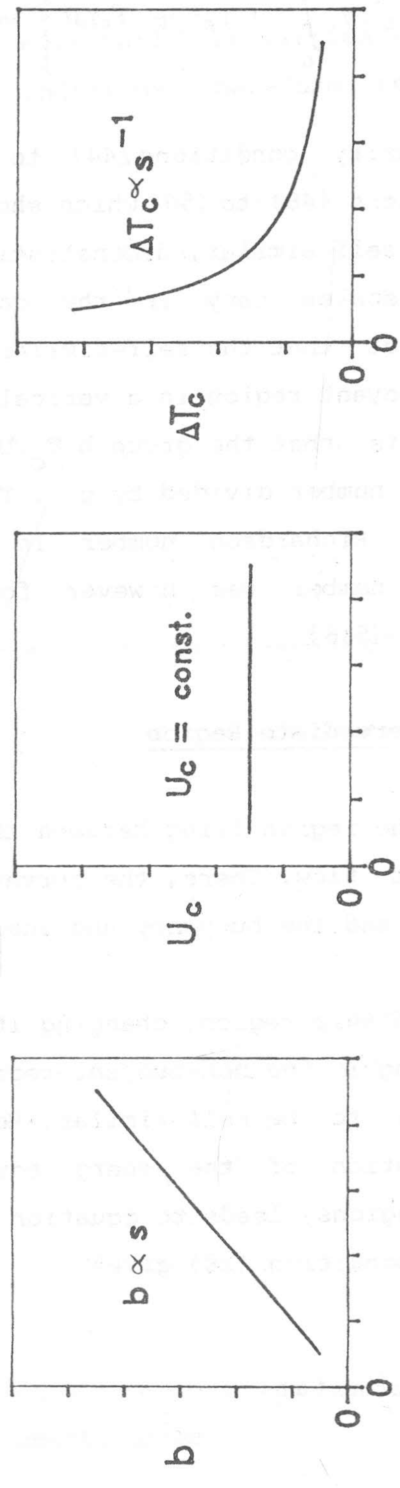
b
0
0
b ∝ s
s
Uc
0
0
Uc = const.
s
ΔTc
0
0
ΔTc ∝ s^-1
s

Fig. 3

Since $\cos\theta$ increases along the jet flow, the above equation requires that U_c^{-3} should also increase in the same direction at a rate equivalent to that of cas θ. This is physically impossible because it obviously contradicts the entrainment phenomenon. This confirms that self-similarity does not exist in the intermediate region.

3. Conclusion

1. Self-similarity analysis, undertaken by the present paper, has shown that the jet flow exhibits self-similarity in both the near field and far field regions. The intermediate region, where the curvature of the jet flow is most significant and the buoyancy and inertia forces are exchanging roles does not show a self-similar behaviour.

2. The different scales of the flow in the near region were found to vary as: $b \propto S$, $U_c \propto s^{-0.5}$ and $T \propto S^{-0.5}$, while for the far region, $b \quad S$, U_c = Constant, and $T_c \propto S^{-1}$.
3. The analysis showed that the effective bulk Richardson number, $gb \quad T_c/\bar{U}_c^2$ is constant through the far (buoyancy dominated) region, a result which coincides with that of the corresponding region of a vertical buoyant jet.