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NUMERICAL SOLUTION OF MULTI-DIMENSIONAL STEFAN PROBLEM

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Abstract

A combination of the method of lines, the iterative alternating directions implicit technique and the invariant imbedding approach is used to obtain the numerical solution of a multi-dimensional free boundary problem, namely, the two dimensional Stefan problem. The analysis of the iterative algorithm is presented and numerical results for a practical situation are given.

1. Introduction

Free or moving surface problems are those problems where the governing equations must be solved suject to certain boundary conditions specified on an a priori unknown surface. Hence the determination of such surface is itself part of the solution of the given problem. Many applications lead to such free boundary value problems, for example: wave propagation, flow through porous medium, the ablation process of a solid and many others.

The present paper is concerned with a specific example, namely, that of the ablation process of a solid. Such application can lead to a two dimensional free boundary problem of the Stefan type. There are numerous methods for the numerical solution of such problem. For example, there are methods that use a specific heat capacity to represent the latent heat phase change [1,2]; others are based on invariant imbedding approach [3,4,5], still others use the so-called freezing index [6,7] and solve a variational inequality [8,9]. The approach suggested in this paper is based on the invariant imbedding technique used in [4,5] combined with the alternating direction implicit technique [10,11].

Recently, the combination of the well-known method of lines and invariant imbedding formalism based on a special alternating direction algorithm, namely, the fractional steps splitting of the governing equation, has been applied in [3,4,5] to obtain numerical solution of the two dimensional free boundary Stefan problem with linear and nonlinear source terms. In [5] an analysis of the multi-dimensional invariant imbedding is given. In this paper we suggest a method that combines the method of lines and the invariant imbedding technique in a way similar to that presented in [3,4,5] but we use a different alternating direction formalism. The analysis of the method follows the same lines of [5]. The numerical experiments are performed on different problems and results are compared with those obtained by other authors.

2. Statement Of The First Problem

The mathematical model governing the ablation of a solid occupying a time dependent domain D(t) can be expressed as a two-dimensional Stefan problem subject to free boundary conditions specified on the free part of the boundary of the domain D(t) to be denoted by $\partial D_2(t)$. The fixed part of the boundary of the domain is denoted by $\partial D_1(t)$. Following [3] this model is written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t} = f(x,y,t) , \quad (x,y) \in D(t) \tag{2.1a}$$

subject to the boundary conditions

$$u = g(x,y,t) , \quad (x,y) \in \partial D_1 \cap [\{ x = 0\} \cup \{ y = 0 \}] , \tag{2.1b}$$

$$\frac{\partial u}{\partial n} = 0 , \quad (x,y) \in \partial D_1 \cap [\{ x = \bar{X}\} \cup \{ y = \bar{Y} \}] , \tag{2.1c}$$

$$\left.\begin{aligned} & u = 0 , && (2.1d) \\ & \nabla u = - \left(g_1 (x,y,\frac{\partial x}{\partial t},t), \; g_2 (x,y, \frac{\partial y}{\partial t} , t)\right) && (2.1e) \end{aligned}\right\} \quad (x,y) \in \partial D_2(t) ,$$

with the initial conditions

$$u(x,y, 0) = u_0 (x,y), \quad (x,y) \in D(0) , \tag{2.1f}$$

and D(0) to be given.

The domain $D(t) \subset (0,\bar{X}) \times (0,\bar{Y})$ where $\bar{X}$ and $\bar{Y}$. are the upper bounds of x and y respectively. Figure 1 gives a graphical description of the above problem. It is assumed that the free boundary can be expressed as $x = \tilde{s}(y,t)$ and has the inverse representation $y = s(x,t)$ which implies the parametric representation ,

$$\partial D_2(t) = \{\tilde{s}(y,t),\ y\},$$

$$\text{or } \partial D_2(t) = \{x,\ s(x,t)\} .$$

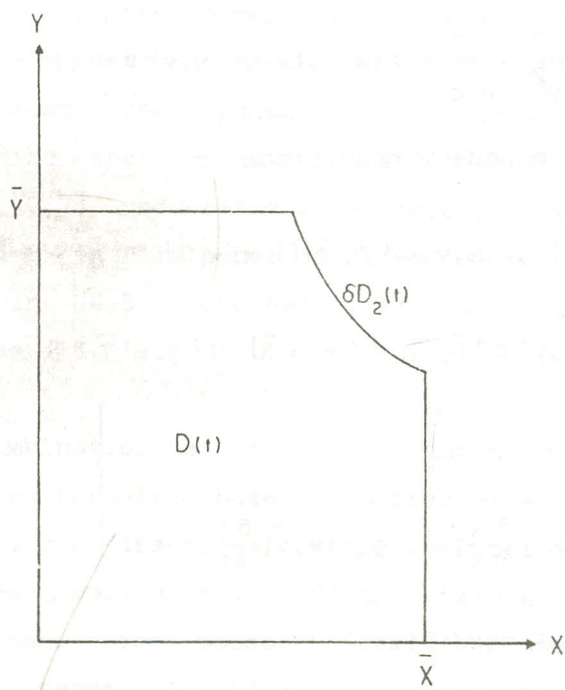

FIGURE 1 - GRAPHICAL DESCRIPTION OF THE DOMAIN UNDER CONSIDERATION.

variable x.

Similarly for the subsequent time interval $t \in [\, t_n + \frac{\Delta t}{2},\ t_n + \Delta t \,]$ we have similar equations where the space variable y is kept continuous while descritization is made with respect to x and t. Namely, we have

$$\ddot{u}_i + \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = f_i(y, t_{n+1}) + \frac{u_i - \tilde{u}_i}{(\Delta t/2)} \qquad (2.3a)$$

subject to

$$u_o(y) = g_o(y, t_{n+1}),\ u_i(0) = g_i(0, t_{n+1}) \qquad (2.3b)$$

$$\dot{u}_i(\bar{Y}) = 0 \qquad (2.3c)$$

$$u_i(s_i) = 0 \qquad (2.3d)$$

$$\dot{u}_i(s_i) = -g_2(x_i, s_i\ \frac{s_i - \bar{s}_i}{(\Delta t/2)},\ t_{n+1}) \qquad (2.3e)$$

where the dot represents differentiation with respect to y,

and $u_i \cong u(x_i, y, t_{n+1})$, $\tilde{u}_i = u(x_i, y, t_{n+\frac{1}{2}})$,

$g_i(y, t_{n+1}) = g(x_i, y, t_{n+1})$,

$s_i = s(x_i, t_{n+1})$, $\bar{s}_i = s(x_i, t_{n+\frac{1}{2}})$.

Thus, along each line $y = y_j$ the solution $\{\tilde{u}_j, \tilde{s}_j\}$ of the multi-point free boundary problem (2.2) must be found for the time interval from t_n to $t_{n+\frac{1}{2}}$. Then subsequently for the time interval from $t_{n+\frac{1}{2}}$ to t_{n+1} the solution $\{u_i, s_i\}$ for the second multi-point free boundary problem (2.3) along each line $x = x_i$ must be found.

The next step is to apply the invariant imbedding approach used in [3] to each locally one-dimensional problem. Accordingly, we consider the Riccati transformation:

$$\tilde{u}_j = \tilde{R}(x)\,\tilde{u}'_j + \tilde{W}_j \qquad (2.4)$$

for the first problem. Following the invariant imbedding formalism we substitute (2.4) into (2.2a), where $\tilde{R}$ and $\tilde{W}_j$ are found from the *initial value problems*

$$\tilde{R}'(x) = 1 - \left(\frac{2}{\overline{\Delta y}^2} + \frac{2}{\Delta t}\right).\tilde{R}^2 \ , \quad \tilde{R}(0) = 0 \ , \qquad (2.5)$$

$$\tilde{W}'_j = -\left(\frac{2}{\overline{\Delta y}^2} + \frac{2}{\Delta t}\right).\tilde{R}(x)\tilde{W}_j + \tilde{R}\left(\frac{2\tilde{u}_j}{\Delta t} + \frac{\tilde{u}_{j-1} + \tilde{u}_{j+1}}{\overline{\Delta y}^2} - f_j(x, t_{n+\frac{1}{2}})\right),$$

$$\tilde{w}_j(0) = g_j\,(0, t_{n+\frac{1}{2}}). \qquad (2.6)$$

The two initial conditions in (2.5) and (2.6) are results of (2.4) and (2.2b). The solution of equation (2.5) is given explicity as

$$\tilde{R}(x) = \frac{1}{\sqrt{\frac{2}{\overline{\Delta y}^2} + \frac{2}{\Delta t}}}.\tanh\sqrt{\frac{2}{\overline{\Delta y}^2} + \frac{2}{\Delta t}}.\,x \ . \qquad (2.7)$$

The numerical solution of (2.6) can be obtained by using a suitable ordinary differential equation solver such as the fourth order Runge-Kutta method. It should be noticed that (2.6) involves $\tilde{u}_{j+1}$, that is the solution along the line $y = y_{j+1}$, and $\tilde{u}_{j-1}$, that is the solution along a previously swept line $y = y_{j-1}$. It was found numerically very statisfactory to use a Gauss-Seidel type of iteration. Hence if a guess $u_j^{(o)}$ is supplied then equation (2.6) for the iterative step k can be conveniently written as

$$\tilde{W}'_j = -\left(\frac{2}{\overline{\Delta y}^2} + \frac{2}{\Delta t}\right).\tilde{R}.\ \tilde{W}_j + \tilde{R}\left(\frac{2}{\Delta t}\ \bar{u}_j + \frac{\tilde{u}_{j-1}^{(k)} + \tilde{u}_{j+1}^{(k-1)}}{\overline{\Delta y}^2} - f_j(x,t_{n+\frac{1}{2}})\right), \tag{2.8}$$

$$\tilde{W}_j(0) = g_j(0,\ t_{n+\frac{1}{2}}). \tag{2.9}$$

Once the values of $\tilde{R}$ and $\tilde{W}_j$ at points along the line $y = y_j$ are found from (2.7) and (2.8) respectively, the position of the free surface along that line ($x = \tilde{s}_j$) is obtained by substituting (2.4) into (2.2e) and using the condition (2.2d). The resulting equation is a nonlinear one and its solution is done iteratively to obtain $\tilde{s}_j^{(k)}$. We have

$$\tilde{\Phi}_j \equiv \tilde{W}_j(\tilde{s}_j^{(k)}) - \tilde{R}(\tilde{s}_j).\ g_1\left(\tilde{s}_j^{(k)},\ y_j,\ \frac{\tilde{s}_j^{(k)} - \bar{\tilde{s}}_j}{(\Delta t/2)},\ t_{n+\frac{1}{2}}\right) \equiv 0\ .$$

For our purpose it is sufficient for the moment to interpolate between successive points along the line $y = y_j$ between which $\tilde{\Phi}_j$ given above changes sign. If no root is found we put $\tilde{s}_j^{(k)} = \bar{X}$. Once the location of the boundary is determined then we solve the two point value problem

$$\tilde{u}''^{(k)}_j + \frac{\tilde{u}_{j+1}^{(k-1)} - 2\tilde{u}_j^{(k)} + \tilde{u}_{j-1}^{(k)}}{\overline{\Delta y}^2} = f_j(x,t_{n+\frac{1}{2}}) + \frac{\tilde{u}_j^{(k)} - \bar{u}_j}{(\Delta t/2)} \tag{2.10}$$

subject to (2.2b) and (2.2e) at both ends which are now known. This process is repeated for all the lines $y=y_j$, $j = 1,2,\ldots, N$ where $y_N = \bar{Y}$. It is evident that along $y = 0$ for $j = 0$ the solution is given by the first equation in (2.2b).

Secondly the corresponding equations resulting from the application of the invariant imbedding to the second problem (2.3) are treated in exactly the same manner described above for problem (2.2). The analogy between the resulting equations and (2.4) - (2.10) is obvious.

At this stage we have two sequences of solutions $\{\tilde{u}_j^{(k)}, \tilde{s}_j^{(k)}\}_{j=0}^{N}$ and $\{u_i^{(k)}, s_i^{(k)}\}_{i=0}^{M}$ and we say that a new cycle of the iterative process has been completed. Convergence is considered achieved if the maximum relative error taken over both sequences and in both space directions is less than a pre-specified tollerance; otherwise a new cycle has to be started. Once convergence is achieved a new time step is considered.

2.2 Numerical Results

For the above problem we choose $f(x,y,t) = 0$, $D(0) = \{[0,1] \times [0,1] -$

$$\{(x,y): (x-1)^2 + (y-1)^2 \leq (\tfrac{1}{4})^2\}\}.$$

The source terms are chosen as

$$g_1 = \frac{1-x}{\left[(1-x)^2 + (1-y)^2\right]^{3/2}} + \frac{dx}{dt},$$

$$g_2 = \frac{1-x}{\left[(1-x)^2 + (1-y)^2\right]^{3/2}} + \frac{dy}{dt}.$$

numerically very statisfactory to use a Gauss-Seidel type of iteration. Hence if a guess $u_j^{(o)}$ is supplied then equation (2.6) for the iterative step k can be conveniently written as

$$\tilde{W}'_j = -\left(\frac{2}{\bar{\Delta y}^2} + \frac{2}{\Delta t}\right).\tilde{R}.\ \tilde{W}_j + \tilde{R}\left(\frac{2}{\Delta t}\ \bar{u}_j + \frac{\tilde{u}_{j-1}^{(k)} + \tilde{u}_{j+1}^{(k-1)}}{\bar{\Delta y}^2} - f_j(x, t_{n+\frac{1}{2}})\right), \tag{2.8}$$

$$\tilde{W}_j(0) = g_j(0,\ t_{n+\frac{1}{2}}). \tag{2.9}$$

Once the values of $\tilde{R}$ and $\tilde{W}_j$ at points along the line $y = y_j$ are found from (2.7) and (2.8) respectively, the position of the free surface along that line ($x = \tilde{s}_j$) is obtained by substituting (2.4) into (2.2e) and using the condition (2.2d). The resulting equation is a nonlinear one and its solution is done iteratively to obtain $\tilde{s}_j^{(k)}$. We have

$$\tilde{\Phi}_j \equiv \tilde{W}_j(\tilde{s}_j^{(k)}) - \tilde{R}(\tilde{s}_j).\ g_1\left(\tilde{s}_j^{(k)},\ y_j,\ \frac{\tilde{s}_j^{(k)} - \bar{\tilde{s}}_j}{(\Delta t/2)},\ t_{n+\frac{1}{2}}\right) \equiv 0$$

For our purpose it is sufficient for the moment to interpolate between successive points along the line $y = y_j$ between which $\tilde{\Phi}_j$ given above changes sign. If no root is found we put $\tilde{s}_j^{(k)} = \bar{X}$. Once the location of the boundary is determined then we solve the two point value problem

$$\tilde{u}''^{(k)}_j + \frac{\tilde{u}_{j+1}^{(k-1)} - 2\tilde{u}_j^{(k)} + \tilde{u}_{j-1}^{(k)}}{\bar{\Delta y}^2} = f_j(x, t_{n+\frac{1}{2}}) + \frac{\tilde{u}_j^{(k)} - \bar{u}_j}{(\Delta t/2)} \tag{2.10}$$

As in [4] the elimination of dx/dt and dy/dt leads to the following equations that describe the motion of the free boundary in x and y directions respectively:

$$\frac{\partial x}{\partial t} = -\left[1+\left(\frac{\partial \tilde{s}}{\partial y}\right)^2\right]\frac{\partial \tilde{u}}{\partial x} + \left[\frac{1}{(1-x)^2+(1-y)^2}\right]^{3/2}\left[(1-y)\frac{\partial \tilde{s}}{\partial y} - (1-x)\right],$$

and

$$\frac{\partial y}{\partial t} = -\left[1+\left(\frac{\partial s}{\partial y}\right)^2\right]\frac{\partial u}{\partial y} + \left[\frac{1}{(1-x)^2+(1-y)^2}\right]^{3/2}\left[(1-x)\frac{\partial s}{\partial x} - (1-y)\right].$$

As for the initial and boundary conditions we take $u_o = 1$ on $\bar{D}(0)$, $u_o = 0$ otherwise, and $g \equiv 1$. The propagation of the free boundary is displayed in Figure 2. numerical results are in good agreement with those obtained in [4] for the same problem.

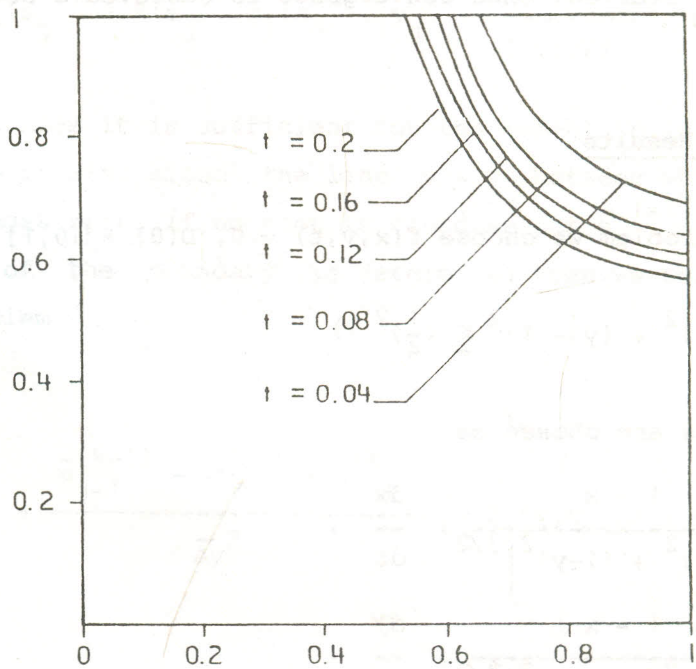

FIGURE 2 - EVOLUTION OF THE FREE BOUNDARY FOR THE FIRST PROBLEM ,

$\Delta x = \Delta y = 0.02$, $\Delta t = 0.002$

3. Statement of The Second Problem

In the present case we consider the numerical solution of a two-dimensional Stefan problem [4] with a known analytical solution. Let us consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t} = f(x,y,t) ,$$

$$u = g(x,y,t), \quad (x,y) \in \partial D_1 (t)$$

$$u = 0 , \quad (x,y) \in \partial D_2(t)$$

$$u = (g_1 - \frac{\partial x}{\partial t}, g_2 - \frac{\partial y}{\partial t}), \quad (x,y) \in D_2(t),$$

where f, g, g_1 and g_2 are chosen such that $u = t(t-x-y)$ is the exact solution of the above problem. The free surface is the line $t-x-y = 0$.

3.1 Numerical Results

The evolution of the free surface of the above problem is investigated by the numerical algorithm described in this paper. Figure 3 shows plots of the free surface. Results are in agreement with the anaytical expressions and with numerical results presented in [4] as well.

4. Analysis of The Method

In this section we consider the question of convergence of the method described in the previous sections. The analysis is carried on following the same outlines presented in [5]. The same results are obtained for the present alternating direction formalism. For the sake

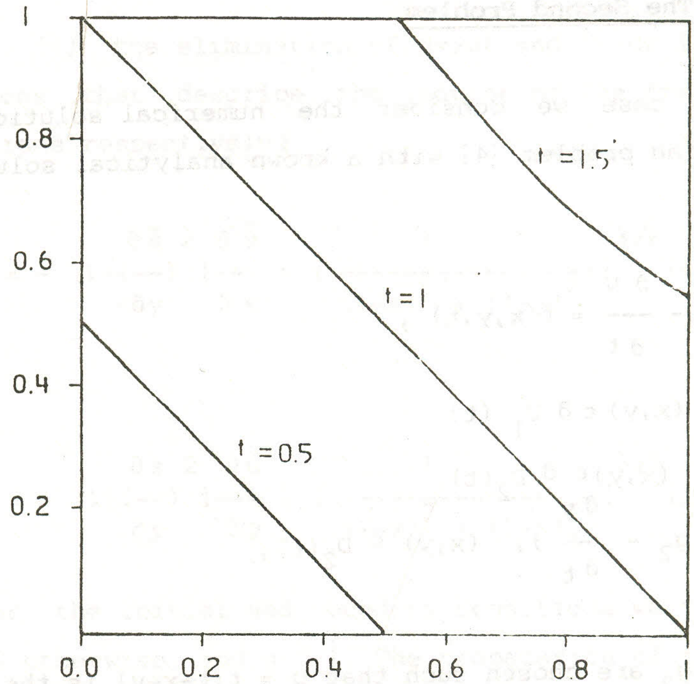

FIGURE 3 - EVOLUION OF THE FREE SURFACE FOR THE SECOND PROBLEM, $\Delta x = \Delta y = 0.02$, $\Delta t = 0.05$

of simplicity of the analysis we consider the following model problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t} = f\,(x,y,t), \quad (x,y) \in D(t) \qquad (4.1a)$$

$$u = g\,(x,y,t), \qquad (x,y) \in \partial D_1(t), \qquad (4.1b)$$

$$u = \frac{\partial u}{\partial n} = 0, \qquad (x,y) \in \partial D_2(t). \qquad (4.1c)$$

where $\frac{\partial u}{\partial n}$ denotes the differentiation along the outward normal direction to the boundary. And initially at $t = 0, u = u_0(x,y)$ and $D(0)$ are given.

Following the method suggested above we obtain the following discretized version of the model problem (4.1):

$$\tilde{u}_j'' + \frac{\tilde{u}_{j+1} - 2\tilde{u}_j + \tilde{u}_{j-1}}{\overline{\Delta y}^2} = F_1\,(x, y_j, t_{n+\frac{1}{2}}, \tilde{u}_j, \bar{u}_j),$$

$$\tilde{u}_j(0) = g(0, y_j, t_{n+\frac{1}{2}}), \qquad (4.2b)$$

$$\tilde{u}_j(\tilde{s}_j) = \tilde{u}_j'(\tilde{s}_j) = 0, \qquad (4.2c)$$

where $F_1 = f(x, y_j, t_{n+\frac{1}{2}}) + \dfrac{\tilde{u}_j - \bar{u}_j}{(\Delta t/2)}$.

Similarly :

$$\ddot{u}_i + \frac{u_{i+1} - 2u_i + u_{i-1}}{\overline{\Delta x}^2} = F_2\,(x_i, y, t_{n+1}, u_i, \tilde{u}_i),$$

$$u_i(0) = g(x_i, o, t_{n+1}), \qquad (4.3b)$$

$$u_i(s_i) = \dot{u}_i\,(s_i) = 0\ , \qquad (4.3c)$$

where $F_2 = f(x_i, y, t_{n+1}) + \dfrac{u_i - \tilde{u}_i}{(\Delta t/2)}$.

We shall be interested in the analysis of the algorithm for the determination of non-negative solutions of problem (4.1). The corresponding conditions to ensure such solution are given in [5] and are readily extended to the present situation. These conditions are:

i) F_ν and g are continuously differentiable on
$\bar{R} \times \{u : u \geq 0\}$ for $\nu = 1,2$ and ∂R respectively.

ii) $\frac{\partial F_1}{\partial \tilde{u}}$ and $\frac{\partial F_2}{\partial u} \geq a_o > -\lambda_o$ where λ_o is the eigenvalue of the Laplacian operator.

iii) $\sup |F_\nu| < \infty$ for $\nu = 1,2$ where supermum is taken over $R \times \{\tilde{u} : \tilde{u} \geq 0\}$ and $R \times \{u : u \geq 0\}$ respectively.

iv) $g(x,y,t) \geq 0$ on ∂R, $t \geq 0$.

v) $\max\{g(0,y,0), g(x,0,0), -F_1(0,y,0), -F_2(x,0,0)\} > 0$,
$x \in (0,\bar{X})$, $y \in (0,\bar{Y})$.

Under these conditions the following corresponding results [5] are readily obtained:

Result (1) The fixed boundary value problem

$$\tilde{\psi}_j'' + \frac{\tilde{\psi}_{j+1} - 2\tilde{\psi}_j + \tilde{\psi}_{j-1}}{\overline{\Delta y}^2} - a_o \tilde{\psi}_j = \alpha_j(x), \quad j = 1,2,\ldots,N,$$

$$\tilde{\psi}_j(0) = \tilde{\psi}_o(\bar{X}) = 0,$$

$$\tilde{\psi}_o(x) = 0,$$

for $\alpha_j \in C^o(0,\bar{X})$ has a unique solution for sufficiently small Δy. If $\alpha_j \leq 0$ then this solution is non-negative.

An analogous result is also valid for the corresponding fixed boundary problem:

$$\ddot{\psi}_i + \frac{\psi_{i+1} - 2\psi_i + \psi_{i-1}}{\overline{\Delta x}^2} - a_o \psi_i = \alpha_i(y), \quad i = 1,2,\ldots, M,$$

$$\psi_i(0) = \psi_o(\bar{Y}) \quad ,$$

$$\psi_o(y) = 0 \ , \alpha_i \in C^o(0, \bar{Y}) \quad , \alpha_i \leq 0 \ .$$

The proof is given in [5]. The solution $\{\tilde{u}_j, \tilde{s}_j\}$ and $\{u_i, s_i\}$ of problem (4.2) and (4.3) respectively are generated by a Gauss-Seidel iteration process, namely,

$$\tilde{u}_j''^{(k)} - \left[\frac{2}{\overline{\Delta y}^2} + \frac{2}{\Delta t}\right] \tilde{u}_j^{(k)} = \tilde{F}_j^{(k)}(x, t_{n+\frac{1}{2}}), \tag{4.4a}$$

$$\tilde{u}_j^{(k)}(0) = g_j(0, t_{n+\frac{1}{2}}), \tag{4.4b}$$

$$\tilde{u}_j^{(k)}(\tilde{s}_j^{(k)}) = \tilde{u}_j'^{(k)}(\tilde{s}_j^{(k)}) = 0, \tag{4.4c}$$

where

$$\tilde{F}_j^{(k)}(x, t_{n+\frac{1}{2}}) = f(x, y_j, t_{n+\frac{1}{2}}) - \frac{\bar{u}_j}{(\Delta t/2)} - \frac{\tilde{u}_{j-1}^{(k)} + \tilde{u}_{j+1}^{(k-1)}}{\overline{\Delta y}^2} \ .$$

And similar for the second direction we have

$$\ddot{u}_i^{(k)} - \left[\frac{2}{\overline{\Delta x}^2} + \frac{2}{\Delta t}\right] u_i^{(k)} = F_i^{(k)}(y, t_{n+1}), \tag{4.5a}$$

$$u_i^{(k)}(0) = g_i(0, t_{n+1}), \tag{4.5b}$$

$$u_i^{(k)}(s_i^{(k)}) = \dot{u}_i^{(k)}(s_i^{(k)}) = 0 \ . \tag{4.5c}$$

i) F_ν and g are continuously differentiable on $\bar{R} \times \{u : u \geq 0\}$ for $\nu = 1,2$ and ∂R respectively.

ii) $\frac{\partial F_1}{\partial \tilde{u}}$ and $\frac{\partial F_2}{\partial u} \geq a_o > -\lambda_o$ where λ_o is the eigenvalue of the Laplacian operator.

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iv) $g(x,y,t) \geq 0$ on ∂R, $t \geq 0$.

v) $\max\{g(0,y,0), g(x,0,0), -F_1(0,y,0), -F_2(x,0,0)\} > 0$, $x \in (0,\bar{X})$, $y \in (0,\bar{Y})$.

Under these conditions the following corresponding results [5] are readily obtained:

Result (1) The fixed boundary value problem

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$$\tilde{\phi}_j(0) = \tilde{\phi}_o(\bar{X}) = 0 ,$$

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for $\alpha_j \in C^o(0,\bar{X})$ has a unique solution for sufficiently small Δy. If $\alpha_j \leq 0$ then this solution is non-negative.

An analogous result is also valid for the corresponding fixed boundary problem:

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$$\psi_i(0) = \psi_o(\bar{Y}) \quad ,$$

$$\psi_o(y) = 0 \;, \alpha_i \in C^o(0, \bar{Y}) \quad , \alpha_i \leq 0 .$$

The proof is given in [5]. The solution $\{\tilde{u}_j, \tilde{s}_j\}$ and $\{u_i, s_i\}$ of problem (4.2) and (4.3) respectively are generated by a Gauss-Seidel iteration process, namely,

$$\tilde{u}_j''^{(k)} - \left[\frac{2}{\overline{\Delta y}^2} + \frac{2}{\Delta t}\right] \tilde{u}_j^{(k)} = \tilde{F}_j^{(k)}(x, t_{n+\frac{1}{2}}), \tag{4.4a}$$

$$\tilde{u}_j^{(k)}(0) = g_j(0, t_{n+\frac{1}{2}}), \tag{4.4b}$$

$$\tilde{u}_j^{(k)}(\tilde{s}_j^{(k)}) = \tilde{u}_j'^{(k)}(\tilde{s}_j^{(k)}) = 0, \tag{4.4c}$$

where

$$\tilde{F}_j^{(k)}(x, t_{n+\frac{1}{2}}) = f(x, y_j, t_{n+\frac{1}{2}}) - \frac{\bar{u}_j}{(\Delta t/2)} - \frac{\tilde{u}_{j-1}^{(k)} + \tilde{u}_{j+1}^{(k-1)}}{\overline{\Delta y}^2} .$$

And similar for the second direction we have

$$\ddot{u}_i^{(k)} - \left[\frac{2}{\overline{\Delta x}^2} + \frac{2}{\Delta t}\right] u_i^{(k)} = F_i^{(k)}(y, t_{n+1}), \tag{4.5a}$$

$$u_i^{(k)}(0) = g_i(0, t_{n+1}), \tag{4.5b}$$

$$u_i^{(k)}(s_i^{(k)}) = \dot{u}_i^{(k)}(s_i^{(k)}) = 0 \cdot \tag{4.5c}$$

The solutions $\{\tilde{u}_j^{(k)}, \tilde{s}_j^{(k)}\}$ and $\{u_i^{(k)}, s_i^{(k)}\}$ are found as described in the previous sections

The existence of a positive solution $\tilde{R}$ and R of the Riccati transformation of the form (2.4) on $(0,\bar{X})$ and $(0,\bar{Y})$ respectively follows from their solutions which are of the type given in (2.7). The other two equations for $\tilde{W}_j$ and W_i are linear and have bounded solutions for bounded $\tilde{F}$ and F respectively; which is guaranteed by the assumptions given above.

It should be mentioned that if $\tilde{s}_j^{(k)} < \bar{X}$ then we set $\tilde{u}_j^{(k)}=0$ on $[\tilde{s}_j^{(k)}, \bar{X}]$ and similar extension is used if $s_i^{(k)} < \bar{Y}$.

Result (2) Let $\tilde{u}_j^{(o)}(0) \geq 0$ for $j = 1,2,\ldots,N$ then $\tilde{s}_j^{(k)}>0$ and $\tilde{u}_j^{(k)} > 0$ on $(0,\tilde{S}_j^{(k)})$ for $j = 1,2,\ldots, N$ and $k=1,2,\ldots$.

PROOF From the given assumptions on g and F_1 it follows that $\tilde{W}_j''(0) = g(0,y_j, t_{n+\frac{1}{2}}) > 0$ and $\tilde{W}_j''(0) > 0$. From the governing equation of $\tilde{W}_j$ it follows that $\tilde{W}_j > 0$ on $(0, \tilde{s}_j^{(k)})$. If $\tilde{u}_j^{(k)}$ has a relative minimum at $x^* \varepsilon (0,\tilde{s}_j^{(k)})$ then $\tilde{u}_j'(x^*) = 0$ and hence from the Riccati transformation (2.4) we get $\tilde{u}_j^{(k)}(x^*) = \tilde{W}_j(x^*) > 0$. An analogous result is also valid for $u_i^{(k)}$ and $s_i^{(k)}$ for their corresponding problem.

It can be easily observed that if $\tilde{u}_j^{(k)} - \tilde{u}_j^{(k-1)} \geq 0$ for all j and k preceeding the calculations along the line $y = y_m$ in iteration step ℓ then

$$\tilde{F}_m^{(\ell)}(x, t_{n+\frac{1}{2}}) - \tilde{F}_m^{(\ell-1)}(x,t_{n+\frac{1}{2}}) \leq 0 \quad .$$

This is easily established by recalling the definition of $\tilde{F}$ and noticing that $|\partial \tilde{F}/\partial \tilde{u}| = (2/\Delta t)$. Similar observation is also valid for the second problem.

Result (3) Let $\tilde{u}_j^{(o)} = 0$, $\tilde{s}_j^{(o)} = 0$ for $j = 1,2,\ldots, N$,

then $\tilde{u}_m^{(\ell)} \geq \tilde{u}_m^{(\ell-1)}$ and $\tilde{s}_m^{(\ell)} \geq \tilde{s}_m^{(\ell-1)}$.

Proof For $\ell = 1$ the result is immediate. Suppose that the assertion is true for all ℓ and m preceeding the claculations along the line $y = y_j$ and iteration k. Then

$$(\tilde{W}_j^{(k)} - \tilde{W}_j^{(k-1)})' = -[\frac{2}{\Delta y^2} + \frac{2}{\Delta t}]R(\tilde{W}_j^{(k)} - \tilde{W}_j^{(k-1)}) - R(\tilde{F}_j^{(k)} - \tilde{F}_j^{(k-1)}),$$

$$(\tilde{W}_j^{(k)}(0) - \tilde{W}_j^{(k-1)}(0)) = 0.$$

Hence by the above observation we get $\tilde{W}_j^{(k)} \geq \tilde{W}_j^{(k-1)}$ and hence $\tilde{s}_j^{(k)} \geq \tilde{s}_j^{(k-1)}$.

Furthermore the maximum principle leads to $\tilde{u}_j^{(k)} \geq \tilde{u}_j^{(k-1)}$.

Similarly analogous results for u_i and s_i can be concluded. The previous results show that the sequences $\{\tilde{u}_j^{(k)}, \tilde{s}_j^{(k)}\}$ and $\{u_i^{(k)}, s_i^{(k)}\}$ have monotonically increasing lower bounds.

Similarly it can be shown that they also have monotonically decreasing upper bounds.

Let $\tilde{\Gamma}_j = \tilde{u}_j + \max_{\partial R} g\,(x,y,t_{n+\frac{1}{2}})$ and let $\tilde{L}$ be the operator acting on $\tilde{u}_j^{(k)}$ in (4.4a), then the following results are obtained:

Result (4) Let $\tilde{U}_j^{(o)} = \tilde{\Gamma}_j, \tilde{S}_j^{(o)} = \bar{X}$. Then the Gauss-Seidel

iterates $\{\tilde{U}_j^{(k)}, \tilde{S}_j^{(k)}\}$ satisfy $\tilde{U}_j^{(k)} \leq \tilde{U}_j^{(k-1)}$, $\tilde{S}^{(k)} \leq \tilde{S}^{(k-1)}$.
The proof is done by induction and using the maximum principle for the operator $\tilde{L}$.

Result (5) For the sequences $\{\tilde{u}_j^{(k)}\}$ and $\{\tilde{U}_j^{(k)}\}$ we have
$0 \leq \tilde{u}_j^{(k)} \leq \tilde{U}_j^{(k)} \leq \tilde{\Gamma}_j$.

Proof Since $\tilde{s}_j^{(o)} < \tilde{S}_j^{(o)}$ and $\tilde{u}_j^{(o)} < \tilde{U}_j^{(o)}$,
then by inductive argument and using result 3 we can directly prove the above statement.
Similar result holds for the second problem.

Finally this last result shows that for each j the sequence $\{\tilde{u}_j^{(k)}\}$ is a uniformly bounded sequence. Now we can state the following result.

Result (6) The sequence $\{\tilde{u}_j^{(k)}, \tilde{s}_j^{(k)}\}$ converges to a solution $\{\tilde{u}_j^*, \tilde{s}_j^*\}$.

PROOF The equation $\tilde{L}\tilde{u}_j^{(k)} = \tilde{F}_j$ where $\tilde{L}$ is the operator in (4.4a) implies that $\tilde{u}_j^{(k)}$ is uniformly bounded on $(0, \tilde{s}_j^{(k)})$ and by extension on $(\tilde{s}_j^{(k)}, \bar{X})$. Thus both $\{\tilde{u}_j^{(k)}\}$ and $\{\tilde{u}'^{(k)}_j\}$ are sequences of uniformly bounded equi-continuous functions so $\tilde{u}_j^{(k)}$ converges monotonically to a continuously differentiable function $\tilde{u}_j^*$ as $k \to \infty$, and at the same time the monotonic sequence $\{\tilde{s}_j^{(k)}\}$ converges to a limit $\tilde{s}_j^*$.

Similar argument can be used to show that $\{\tilde{U}_j^{(k)}, \tilde{S}_j^{(k)}\}$

converges to $\{\tilde{U}_j^*, \tilde{S}_j^*\}$.

To guarantee the uniqueness of solution we impose the restriction [5]

$$F_1 = - \frac{\tilde{u}^*_{j+1} + \tilde{u}^*_{j-1}}{\overline{\Delta y}^2} \geq 0 \quad \text{on} \quad [\tilde{s}^*_j, \bar{X}].$$

Similarly, we establish similar results for the second problem under the corresponding restriction

$$F_2 = - \frac{\bar{u}^*_{i+1} + u^*_{i-1}}{\overline{\Delta x}^2} \geq 0 \text{ on } [s^*_i, \bar{Y}].$$

5. Conclusion

In this paper an algorithm is suggested for the numerical solution of multi-dimensional free boundary Stefan problems. The method is based on the combination of the method of lines and invariant imbedding [5] using the alternating direction implicit iterative formalism. The formulation of the resulting problem falls within the frame of the class of problems considered in [5] which leads to adopting similar arguments for the analysis of the method. The results of the numerical experiments using the suggested method are in good agreement with previously obtained results.

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