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NONLINEAR ANALYSIS OF STIFFENED AND UNSTIFFENED PLATES

Mahmoud H. Metwally* Ph.D and Mohamed Salah El Din**

* Ass. Prof. of Structural Engineering

** Demonstrator

Structural Engineering Department

Faculty of Engineering, Alexandria University

Alexandria, Egypt

Abstract

In order to perform a detailed analysis of large deflection behavior of unstiffened plate and stiffened plate, an alternate finite element formulation is developed and the geometric stiffness matrices are completely formed for nonconforming rectangular element. The formulation is coded into an existing general purpose computer programme for small-deflection analysis, large-deflection analysis and stability analysis.

Example of stiffened plates subjected to in-plane loads is presented. Comparison of results obtained by the present finite element method with those by other methods is made.

Notations

a,b	Length and width of rectangular plate element.
[B]	Strain matrix.
$[B^o]$, $[B^L]$	Linear and non-linear portions of [B].
[D], $[D_m]$, $[D_b]$	Elasticity matrix and its membrane and bending components.
$\{\delta\}$, $\{\delta_m\}$, $\{\delta_b\}$	Vectors of nodal point displacements and its in-plane and bending components.
E	Modulus of elasticity
$\{\varepsilon\}$	The strain vector
$\{\varepsilon_m\}$, $\{\varepsilon_b\}$	Membrane and bending strain vectors
$\{\varepsilon_m^o\}$, $\{\varepsilon_m^L\}$	Linear and non-linear portions of membrane strain
{F}	Internal forces vector.
h	Thickness of the plate
$[K_o]$, $[K_o^m]$, $[K_o^b]$	Small displacement stiffness matrix and its membrane and bending components.
$[K_L]$	Initial displacement stiffness matrix.
$[K_\sigma]$	Initial stress stiffness matrix
$[K_G]$	Geometric stiffness matrix; $= [K_L] + [K_\sigma]$
$[K_T]$	Tangential stiffness matrix; $= [K_o] + [K_G]$
$[K_s]$	Secant stiffness matrix.
ν	Poisson's ratio
{P}	External force vector
$\{\psi\}$	Residual (unblanced) force vector.
$\{\sigma\}$	Stress resultant vector
$\{\sigma_m\}$	$= \{N_x, N_y, N_{xy}\}$, membrane resultant vector.
$\{\sigma_b\}$	$= \{M_x, M_y, M_{xy}\}$, Bending stress resultant vector.
u,v,w	Displcaements in x,y,z directions.

$\theta_x, \theta_y, \theta_z$	Rotations around x,y,z directions.
x,y,z	cartesian coordinates.
V	Volume.

1. Introduction

Behaviour of unstiffened plates and stiffened plates has been a subject of interst for many years. A large amount of research work and parametric studies of the response of those plates are available in the literature. However, due to its complexity and many influencing parameters involved, more understanding of all aspects of the behavior is still desired.

Optimum designs with respect to weight could be obtained in the presence of constraints due to local and general buckling, maximum tensile and compressive stress of strain, maximum shear strain and lower and upper bounds or skin layer thicknesses, stiffener cross section dimensions and stiffener spacings.

Design parameters of stiffened panels, allowed to vary during the optimization phase, include panel skin thickness, spacing of stiffeners and thicknesses and width of the segments of stiffener cross section.

The purpose of this research is to describe the element etiffness matrix in the nonlinear analysis and to obtain accurate predictions for nonlinear behavior of stiffened plate under different types of loadings.

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u,v,w	Displcaements in x,y,z directions.

2. Geometric Non-Linearity

If the deflection of a plate is large that is, on the order of the plate thickness but is small as compared to the other dimensions, then the deformed geometry will obviously differ significantly from the underformed geometry. The deflection of a plate are accompanied by stertching of the middle surface, provided that the edges of the plate are restrained against in-plane motion. If the deflections are small, the membrane stresses produced by such stretching are neglected but, if the deflections are large, these memberane stress can cause a considerable decrease of displacement and can help appreciably in carrying the lateral loads. This results in a nonlinear strain-displacement relationship. Large displacements problems of this type are said to be "geometrically non-linear".

A well known theory of plate bending that includes the effect of middle plane deformations was presented by Von Karman (1) in 1910. Von Karman derived a particularly compact form for the governing equations for isotropic plates, expressing them as two simultaneous, fourth order, differential equations in terms of the lateral deflection and Airy's stress function.

Geometrical non-linearity is feature of "elastic stability" problems which frequently occur in structural mechanics. In this problems, as the reader is aware, a structure becomes unstable and eventually "buckles", if the applied load exceeds its "critical" or "buckling" load. From the design point of view, calculation of the "critical" loads of structures is of considerable importance. In the other side, it has become apparent that the solution of a stability problem usually involves not only the determination of the critical load but also the construction of the nonlinear load-deflection curve for the actual structure.

3. Finite Element Analysis of Geometrically Nonlinear Behavior

The first work on the extension of the finite-element procedure to geometrically nonlinear structure was reported by Turnur et al (2). A linearized incremental analysis procedure was described and so-called geometric stiffness matrixes derived for pin-jointed bar and triangular plane stress elements. Since bending was not included, their approach is restricted to ivestigating instability resulting only from unstable modal configurations. In a subsequent paper, Gallagher and Padlog (3) outlined a consistent procedure, based on the principle of minimum potential energy, for introducing geometric nonlinearity in finite-element displacement method. Their formulation is restricted to a linearized stability analysis, i.e., where the behavior prior to buckling (bifuraction) is linear. This assumption is introduced by neglecting the nonlinear rotational terms in the strain-displacement relations prior to bucling. They also derived a linearized tangent stiffness matrix for a beam-column element. Extensive studies on large deflections of beam, membranes, plates and cylindrical panels have since been carried out in connection with the use of finite elements.

4. Deformation-Displacement Relations

We shall describe the plate strains in term's of middle surface displacements; i.e., if the x-y plane coincides with the middle surface and the middle surface dispalcements are u,v and w in the x,y and z directions, the strain vector $\{\varepsilon\}$ at any point (x,y) is expressed as;

$$\{\varepsilon\} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ ------ \\ -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \frac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \left(\dfrac{\partial w}{\partial x}\right)\left(\dfrac{\partial w}{\partial y}\right) \\ ------ \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_m^o \\ --- \\ \varepsilon_b^o \end{Bmatrix} + \begin{Bmatrix} \varepsilon_m^L \\ --- \\ 0 \end{Bmatrix} \qquad (1)$$

in which the first term is the linear portions of $\{\varepsilon\}$ and the second gives the nonlinear terms and ;

$\{\varepsilon_m^o\}$, $\{\varepsilon_m^L\}$ =Linear, nonlinear vectors of the membrane strain $\{\varepsilon_m\}$ and

$\{\varepsilon_b^o\}$ = middle surface bending deformation vector.

5. Stress-Strain Relations

Define $\{\sigma\}$ as the stress resultant vector corresponding to strain vector $\{\varepsilon\}$ expressed as;

$$\{\sigma\} = \begin{Bmatrix} \sigma_m \\ .... \\ \sigma_b \end{Bmatrix} \qquad (2a)$$

which contains the following components,

$$\text{inplane} \quad \{\sigma_m\} = \{N_x, N_y, N_{xy}\}^T = \{\int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) dz\}^T \qquad (2b)$$

$$\text{bending} \quad \{\sigma_b\} = \{M_x, M_y, M_{xy}\}^T = \{\int_{-h/2}^{h/2} (\sigma_x \cdot \sigma_y, \tau_{xy}) z dz\}^T \qquad (2c)$$

If only linear elastic behavior is considered, we can write the general stress-strain relations as;

$$\{\sigma\} = [D] \quad \{\varepsilon\} \qquad (3)$$

in which, matrix [D] is composed of the usuall in-plane and bending elastic components $[D_m]$ and $[D_b]$ as follow;

$$[D] = \begin{bmatrix} [D_m] & 0 \\ 0 & [D_b] \end{bmatrix} \qquad (4)$$

6. __Rectangular Plate Element__

Since, a non-linear analysis requires several iterations, the time to compute the matrices was important. It was decided that the number of nodes should be minimum. Hence a rectangular element, as shown in Fig. (1), with 4 corner nodes is used in the present plate large deflection analysis. Six degrees of freedom are considered in each nodal point:

(1) Two in-plane displacements u and v in x and y directions, respectively; (2) one transverse deflection w, (3) two rotations θ_x and θ_y about x and y axes respectively; and a generalized twist θ_{xy}

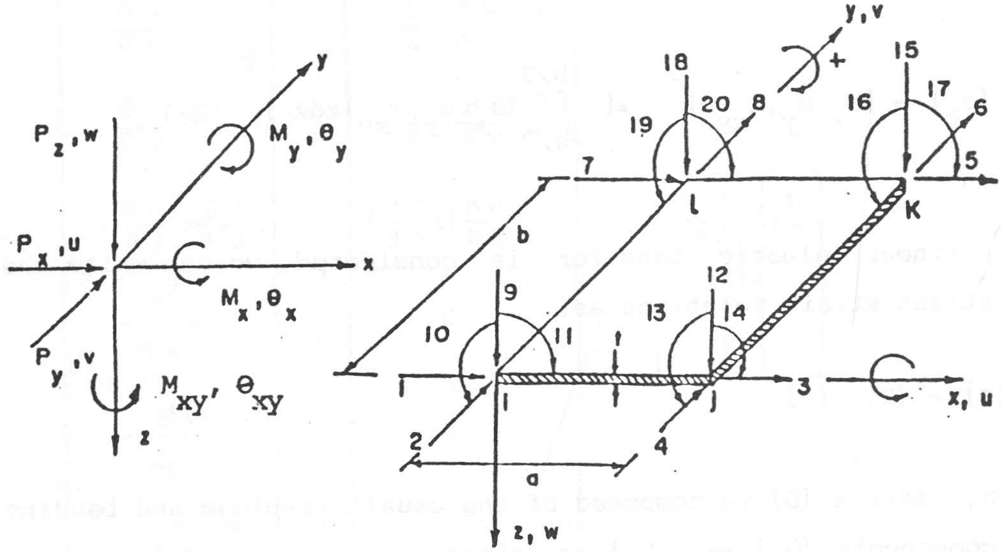


Fig. 1. Joint Forces & Displacements

To avoid numerical integration schemes, when evaluating stiffness terms, the element must have simple displacement functions. The u and v displacement functions are chosen as two polynomials which contain the same number of unknown constants as the number of nodal in-plane displacements. The w displacement function is obtained from a fourth-degree polynomial which has 12 terms that equal to the total number of bending displacements. The u,v and w functions are sympolized as:

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

$$v = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy$$

$$w = \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} x^2 + \alpha_{13} xy + \alpha_{14} y^2 + \alpha_{15} x^3 + \alpha_{16} x^2 y + \alpha_{17} xy^2 + \alpha_{18} y^3 + \alpha_{19} x^3 y + \alpha_{20} xy^3 \quad (5)$$

Note that in the w function, certain terms of the complete fourth-order polynomial are neglected.

Referring to Fig. (1), the membrane and bending displacement vectors for node i are written as:

$$\{\delta_{mi}\} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \quad \text{and} \quad \{\delta_{bi}\} = \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} = \begin{Bmatrix} w_i \\ -\left(\frac{\partial w}{\partial y}\right)_i \\ \left(\frac{\partial w}{\partial x}\right)_i \end{Bmatrix} \quad (6)$$

Finally, we define δ as the element displacement matrix, i.e.

$$\{\delta\} = \begin{Bmatrix} \delta_m \\ \cdots \\ \delta_b \end{Bmatrix} \quad (7)$$

7. Finite Element Stiffness Formulation

It is convenient to return to the basic formulation of the finite element equation from the principles of virtual work. If ψ represents the vector of the sum of the internal and external forcses and by studying internal and external work changes, we can write:

$$d\{\sigma\}^T\{\psi\} = \int_V d\{\varepsilon\}^T \cdot \{\sigma\}\, dV - d\{\delta\}^T \cdot \{P\} = 0 \qquad (8)$$

where V is the undeformed volume,

$d\{\varepsilon\}$ the virtual strain vector due to the virtual displacements $d\{\delta\}$

and $\{P\}$ represents all the external forces

Refere to Eq. (1), the variation of strain can be expressed in terms of virtual displacements as;

$$d\{\varepsilon\} = [B]\, d\{\delta\} \qquad (9)$$

Then, on elimination of $d\{\delta\}^T$ from Eq. (8) we have generally valid equation

$$\{\psi(\delta)\} = \int_V [B]^T \{\sigma\}\, dV - \{P\} = 0 \qquad (10)$$

If the dependance of $\{\sigma\}$ on strain and hence on displacements can be determined, we have therefore to solve a non-linear equation.

$$\{\psi(\delta)\} = \{F(\delta)\} - \{P\} = 0 \qquad (11)$$

in which $\{F(\delta)\}$ represents the internal forces dependent non-linearly on displacements. If displacements are large, the strains depend non-lineary on displacements and the strain matrix [B] is now dependent on $\{\sigma\}$, we can write

$$[B] = [B^o] + [B^L(\delta)] \qquad (12)$$

in which $[B^o]$ is the same matrix as linear infinitesimal strain analysis and only $[B^L]$ depends on the displacements. Taking appropriate variation of Eq. (10) with respect to d(δ). we get;

$$d\{\psi\} = \int_V d[B]^T\{\sigma\}\,dV + \int_V [B]^T d\{\sigma\}\,dV \qquad (13)$$

where {P} is independent of {δ} and d {P} = 0

But we have from Eq. (3); {σ}=[D] {ε} and for linear elastic material

$$d\{\sigma\}=[D]d\{\varepsilon\} =[D]\,[B]d\{\delta\} \qquad (14)$$

and from Eq. (12)

$$d[B] = d[B^L] \qquad (15)$$

Substituting for d {σ} and d[B] from Eq. (14), and (15) into Eq. (13), we have

$$d\{\psi\} = \int_V d[B^L]^T\{\sigma\}dV + (\int_V [B]^T[D][B]dV)d\{\delta\} \qquad (16)$$

Substituting for matrix [B] fro, Eq. (12), Eq. (16) becomes;

$$d\{\psi\} =\int d[B^L]^T\{\sigma\}\,dV + [\int_V [B^O]^T[D][B^O] + [B^O]^T[D][B^L]$$

$$+ [B^L]^T [D][B^O] + [B^L]^T[D][B^L]dV] \qquad (17)$$

or $d\{\psi\} = ([K_O] + [K_L] + [K_\sigma])d\{\delta\} = ([K_O]+[K_G])d(\delta)$

$$= [K_T]d\{\delta\} \qquad (18)$$

in which:

- $[K_o]$ represents the usual small dsiplacements stiffness matrix, i.e.

$$[K_o] = \int_V [B^o]^T [D] [B^o] dV \qquad (19\text{-a})$$

The matrix $[K_L]$ is due to large displacements called "the initial displacement matrix" and is given by

$$[K_L] = \int_V [B^o]^T [D] [B^L] + [B^L]^T [D] [B^o] + [B^L]^T [D] [B^L] dV \qquad (19\text{-b})$$

The matrix $[K_\sigma]$ is a symmetric matrix dependent on stress level. This matrix is known as "initial stress matrix" given through the following equation.

$$\int_V d[B^L]^T \cdot \{\sigma\} \, dV = [K_o] \, d\{\delta\} \qquad (19\text{-c})$$

The matrix $[K_G]$ is called "the geometric matrix" and is given by

$$[K_G] = [K_\sigma] + [K_L] \qquad (20)$$

and matrix $[K_T]$ is total, tangential stiffness, matrix

Stiffness Matrix of small displacements equation (19-a) is defined in any of the well known references. However, the rest of the stiffness Matrices, initial displacement Matrix, initial Stress Matrix (equations 19-b, 19-c) are given at the end of this paper.

8. Evaluation of Internal Forces

In non-linear analysis, we need to evaluate the internal forces which depends non-linearly on displacements.

By studing the internal virtual work, we can write;

$$d\{\delta\}^T.\{F(\delta)\} = \int_V d\{\varepsilon\}^T.\{\sigma\}dV = \int_V d\{\varepsilon\}^T[D].\{\varepsilon\}dV \tag{21}$$

Then, the final form of internal forces is:

$$\{F(\delta)\} = \left[\begin{array}{c:c} [K_o^m] & 0 \\ \hdashline 0 & [K_o^b] \end{array}\right] + \left[\begin{array}{c:c} 0 & \frac{1}{2}[K_L^{bm}]^T \\ \hdashline [K_L^{bm}] & \frac{1}{2}[K_L^b] \end{array}\right] \{\delta\}$$

$$= [K_s].\{\delta\} \tag{22}$$

in which ; $[K_s]$ is an unsymmetric matrix, called "secant stiffness matrix", $[K_o^m]$, $[K_o^b]$ are the submatrices of $[K_o]$ and $[K_L^{bm}]$ and $[K_L^b]$ are the submatrices of $[K_L]$

9. Solution Procedures of Non-linear Equation

While the development of the basic equilibrium equations is straight forward procedure, the solution of these equations is a more difficult task. Success in obtaining an accurate solution often dpends primarily upon the solution procedure that one uses. For the solution of the non-linear equilibrium equation [Eq. (11)], a well-known procedure called "Newton-Raphson Method" have been used.

If initial estimate σ_i for the total displacements gives residual (unblanced) forces $\psi(\sigma_i) = 0$, an improved value σ_{i+1} is obtained by finding:

$$\delta_{i+1} = \delta_i + \Delta\delta_{i+1} \tag{23}$$

both $\Delta\delta_{i+1}$ and the unblanced forces ϕ is sufficiently small. One of the problems detract from the use of this Newton method is the making an initial estimate of the solution. A common means of starting a newton-Raphson solution is to assume $\delta = 0$, so that the nonlinear portions of $[K_T]$ are initially set to zero for the first iteration.

A step by step description follows:

1. Initialize iteration number : δ $i=0$
2. Set $\{\delta_o\} = 0$, then the initial internal forces $\{F_o\} = 0$
3. $i = i + 1$
4. Calculate $[K_T]_{i-1}$
5. Solve for the correction displacements $\{\Delta\delta\}_i$ using

   equation; $[K_T]_{i+1} . \{\Delta\delta\}_i = -\{\phi\}_{i-1}$

Then point A_i can be established and this fixes;

$$\{\delta\}_i = \{\delta\}_{i-1} + \{\Delta\delta\}_i$$

6. Knowing $\{\delta\}_i$, the true secant stiffness matrix $[K_s]_i$, for each element is formed and the internal forces corresponding to point B_i are calculated from;

   $$\{F\}_i = \sum_{n=1}^{n=NE} [K_s]_i^n . \{\delta\}_i^n$$

   and the unblanced force ; $\{\phi\}_i = \{F\}_i - \{P\}$
7. Repeat (3) through (6) until $\{\Delta\delta\}_i$ or /and $\{\phi\}_i$ are sufficiently small. This process will lead to convergence at point A .

10. Formulation of the Stability Problem

At the other end of the spectrum from predicting highly non-linear behavior is an important class of elastic instability problem. The relevant solution objective for this problem class is the prediction of the "critical" or "buckling" load level. The critical loads can be estimated readily within the framework of the linear incremental formulation which was stated as;

$$[K_T]_{i-1} \cdot \Delta \delta_i = \Delta P_i \qquad (26)$$

A basis for the detection of a critical point is the existance of a displacement state, near a known equilibrium position, which is accessible without disturbance of the loading state. The mathematical basis for this interrogation follows immediately from Eq. (26), i.e.

$$[K_T].\{\Delta \delta\} = ([K_o] + [K_G])\{\Delta \delta\} = \{0\} \qquad (27)$$

The existance of a non-trivial solution to this non-linear relation serves to identify a point of instability. If the loads are increased by a factor λ, and by assuming that the displacements vary linearly with applied load level, Eq. (27) can be written as;

$$([K_o] + \lambda [K_1] + \lambda^2 [K_2]).\{\Delta \delta\} = \{0\} \qquad (28)$$

in which,

- $[K_o]$ is the will-known linear stiffness matrix,
- $[K_i]$ is the counterpart of the geometric matrix $[K_G]$ which is indirectly a linear function of the displacements.

and $[K_2]$ is the counterpart of the geometric matrix $[K_G]$ which is indirectly a quadratic function of the displacements.

In Eq. (28), the matrices $[K_1]$ and $[K_2]$ are evaluated at λ =1. Eq. (28) is a quadratic eignvalue equation useful for estimating the critical load which is associated with the smallest root λ_{min} and the associated buckling mode shapes which represent the "eigenvectors" of this equations.

Of course, if the elastic $[K_o]$ solution gives such deformations that the initial displacement matrix $[K_L]$ is identically zero, Eq. (28) reduces to a conventional, linear eigenvalue equation, i.e.;

$$([K_o] + \lambda \; [K_\sigma]). \{\Delta\delta\} = 0 \qquad (29)$$

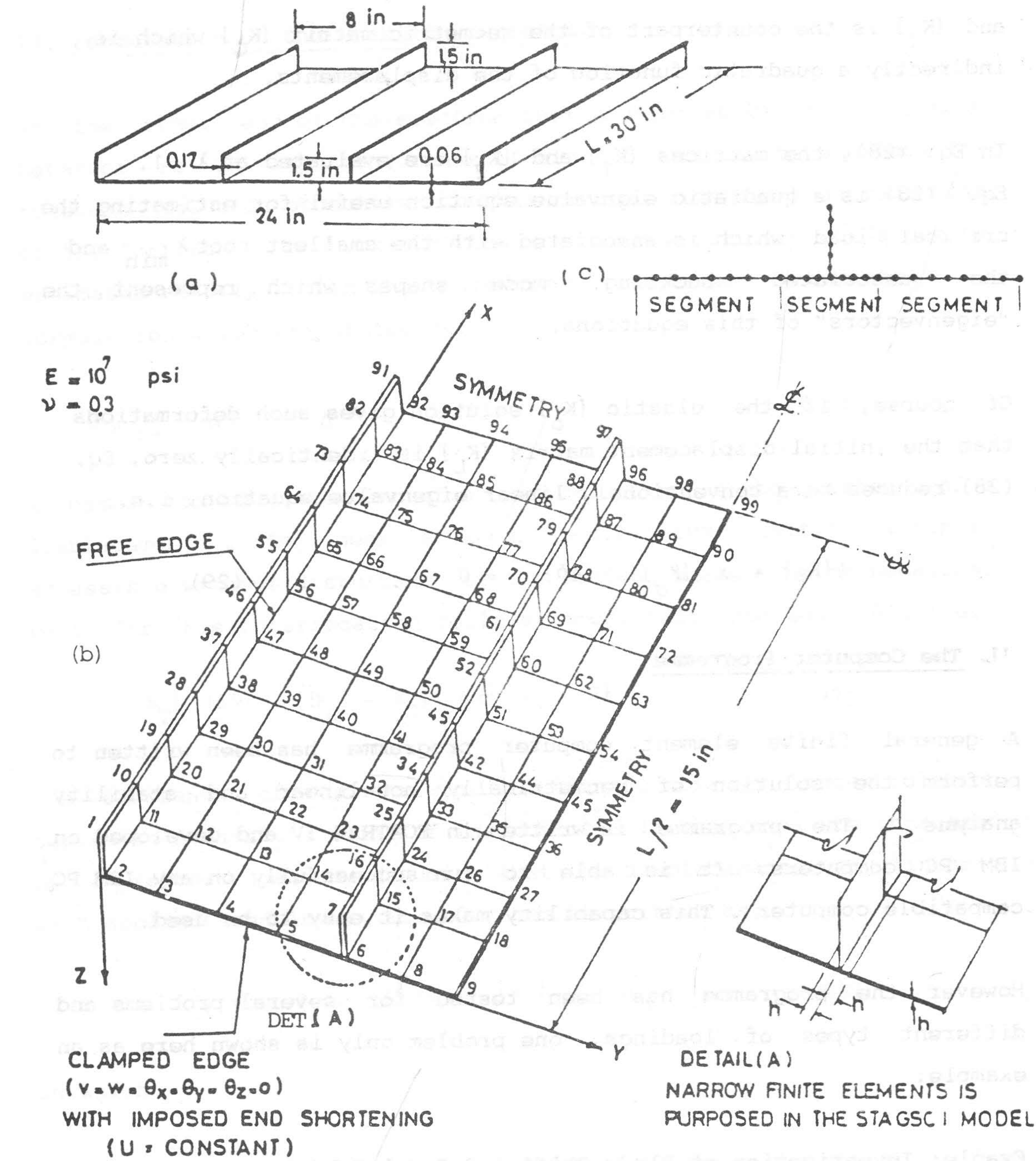
11. The Computer Programme

A general finite element computer programme has been written to perform the solution of geometrically non-linear and stability analysis . The programme is written in FORTRAN IV and developed on IBM PC computers. It is able to run successfully on any IBM PC compatible computer . This capability makes it easy to be used.

However the programme has been tested for several problems and different types of loadings, one problem only is shown here as an example:

12. Exaple; Investigation of Blade-Stiffened Panel Subjected to Uniaxial Compression

A blade-stiffened panel with geometry and linear material, shown in



(Fig 3)
(a) Blade stiffened panel. (b) Discretized model (mesh-1) used for the present analysis.
(c) Panda2 Model

Fig. (3a) was analyzed by STAGSC-1(11) and PANDA2(12). The same panel is analyzed here to compare the results with those obtained with the STAGS and PANDA2 computer programs and to demonstrate many interesting and complex phenomena that occur when the panel is loaded in pure axial compression far into its locally post-buckling regime.

The plate was designed so that it buckles locally between the stringers at an axial load well below the wide column buckling load. It was found that the plate should buckle into five half waves along the entire length. So, one fourth of the panel is included with symmetry planes of midlength and mid-width. Fig. (3b) shows the finite-element mesh, for the present model.

In both the present model and STAGS model, the stiffened edges are free, expect for the presence of stiffener and the loaded edges are subjected to uniform end shortening with no rotation allowed. In PANDA2 analysis, since the plate will buckle locally between stringers, as it was designed, this buckling behavior is assumed to be independant of the boundary conditions along the four panel edges. This is a good assumption, if there are more than two or three half-waves in the local buckling pattern over the length and width of the entire panel.

13. <u>Analysis of Results</u>

(a) critical buckling load

The bifurcation buckling load predicted;

from the present method = 313.6 lb/in.

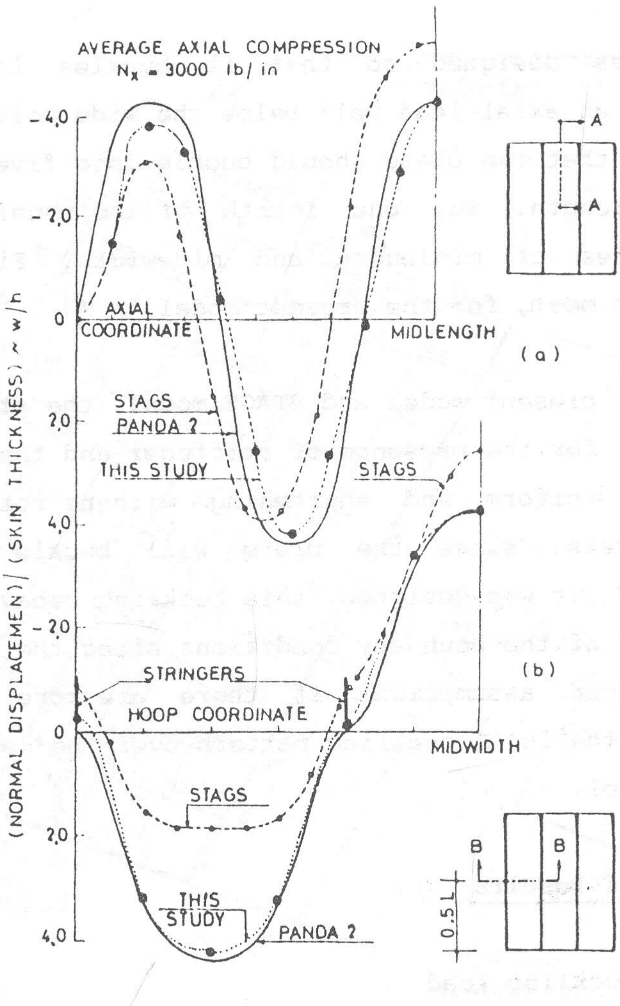


Fig.(6): Normal deflection w shape:
(a) along half the length of the panel in the middle bay, and,
(b) across half the width at the midlength symmetry plane.

from STAGSC-1 = 336.0 lb/in. , and

from PANDA2 = 319.3 lb/in.

(b) Load-deflection curve

Fig. (4) depicts the normal deflection at the center of the panel as a function of normalized axial load. The present method, STAGS and PANDA2 results agree extroadinarily well up to a load factor N_x/N^P_{xcr} = 5.22. At an axial load slightly higher than that, STAGS has difficulty converging. As the axial load is further increased, STAGS predicts maximum normal displacements that exceed those predicted with the present method and PANDA2. The three models predict normal displacements that have equal inward & outward maximum and uniform wave - length, expect, at axial load higher than N_x/N^P_{xcr} = 5.22 the waveness predicted by STAGS becomes more non-uniform.

(c) Load-Axial strain curve

Fig. (5) shows load-axial strain curves from the present method, STAGS and PANDA2. The present method and STAGS show higher axial stiffness because there are four stringers in 24 in. in the present and STAGS models, whereas there are three in the PANDA2 model. At axial load higher than N_x / N^P_{xcr} = 5.22. The present method predicts higher axial stiffness than that predicted by STAGS, because STAGS predicts higher maximum normal displacements than the true values, and consequently this weakens the axial stiffness.

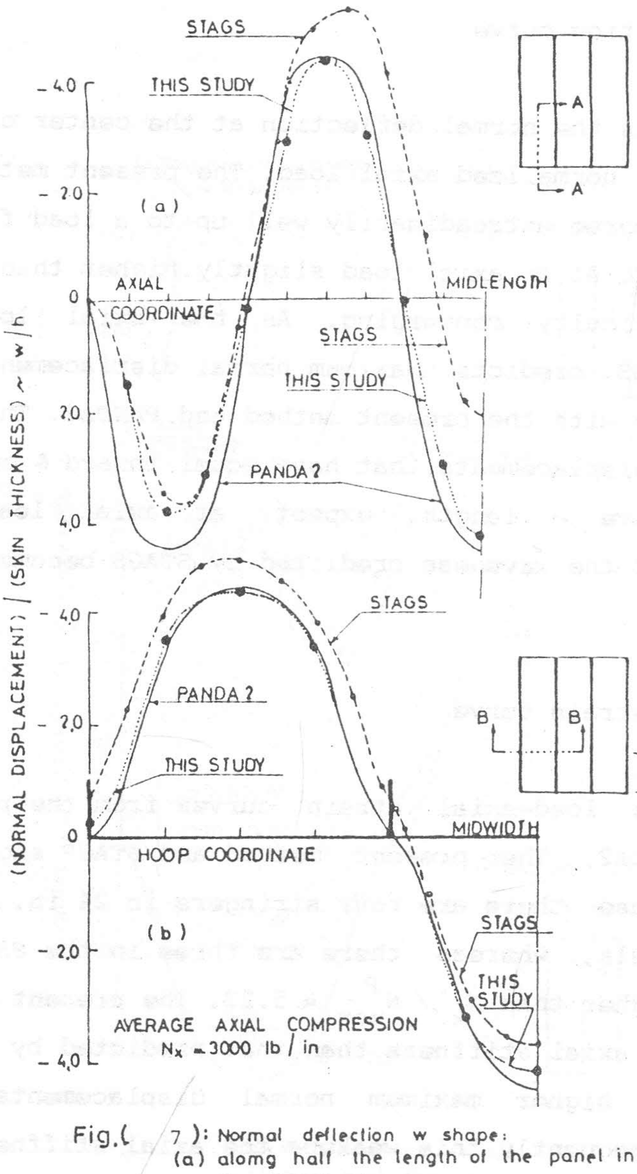
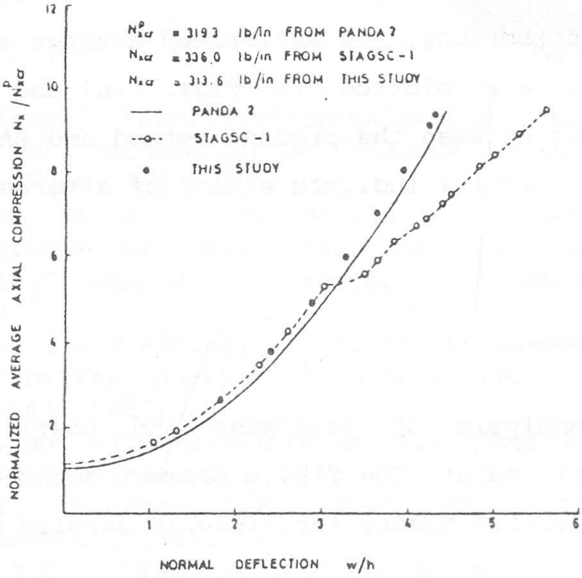


Fig.(7): Normal deflection w shape:
(a) alomg half the length of the panel in one of the side bays, and,
(b) across half the width at distance 0.3L from the loaded edge.



Fig(4) Normal deflection w at the center of the panel

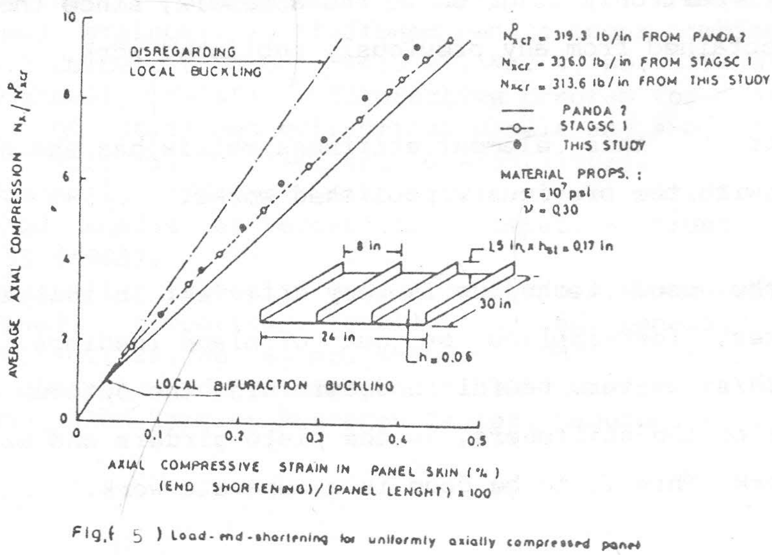


Fig.(5) Load-end-shortening for uniformly axially compressed panel

(d) Deflection distributions:

The present method, STAGS and PANDA2 predictions of normal displacement distributions, at an imposed average axial compression of 3000. lb/in., are plotted in Figs. (6) and (7). Notice that excellent agreement between the present method and PANDA2 is obtained while the STAGS results indicate a kind of alternating discrepancy with them.

14. Conclusion

The non-linear analysis of stiffened and unstiffened plates is presented in this paper. The finite element technique based on Von Karman large deflection theory for linearly elastic material is used in the analysis.

The element stiffness matrix for the non-linear analysis is derived in this work. It is divided into two parts, linear and non-linear. The non-linear part only is given in the appendix, since the linear part could be obtained from any previously published work.

The application of this element stiffness matrix has shown a very good agreement with the previously published works.

The use of the used technique is very efficient in analyzing the stiffened plates, for inplane or out of plane loadings or their combination. This is very useful in determining the optimum spacing and dimensions of the stiffeners, in the plate girders and brackets, in the steel work. This is to be done in a separate work.

Of course, the use of more refined mesh, in the analysis, will produce more accurate results.

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Appendix

1) The initial displacement matrix K_L is given as

$$K_L = \left[\begin{array}{c|c} O & K_L^{bm^T} \\ \hline K_L^{bm} & K_L^{b} \end{array}\right]$$

The submatrices are given in the following tables.

2) The initial stress matrix k_σ is defined as

$$K_a^b = \left[\begin{array}{c|c} O & O \\ \hline O & K_\sigma^b \end{array}\right]$$

K_σ^b is obtained from K_L^b by replacing M,N,L with the internal stresses N_x, N_y, N_{xy} respectively. The whole matrix must be divided by the constant $[Eh/1 - v^2]$.

The First-Order Element Incremental Stiffness Matrix $[K_L^{bm}]$:

$$[K_L^{bm}] = \frac{Eh}{180(1-\nu^2)} \times$$

	1	2	3	4	5	6	7	8	
	(A+J)	(B+I)	-(E-L)	(H-K)	-(C+J)	-(D+I)	(G-L)	-(F-K)	9
	-(U+P)b	-(R-Θ)b	(U-N)b	(R-M)b	(W+N)b	(T+M)b	-(W-P)b	-(T+Θ)b	10
	(Q-P)a	(V+Θ)a	(S+P)a	(Z-Θ)a	-(S+N)a	-(Z+M)a	-(Q-N)a	-(V-M)a	11
	-(E+L)	(H+K)	(A-J)	(B-I)	(G+L)	-(F+K)	-(C-J)	-(D-I)	12
	(U+N)b	(R+M)b	-(U-P)b	-(R+Θ)b	-(W+P)b	-(T-Θ)b	(W-N)b	(T-M)b	13
	-(S-P)a	-(Z-Θ)a	-(Q+P)a	-(V-Θ)a	(Q+N)a	(V+M)a	(S-N)a	(Z-M)a	14
	-(C+J)	-(D+I)	(G-L)	-(F-K)	(A+J)	(B+I)	-(E-L)	(H-K)	15
	-(W+N)b	-(T+M)b	(W-P)b	(T+Θ)b	(U+P)b	(R-Θ)b	-(U-N)b	-(R-M)b	16
	(S+N)a	(Z+M)a	(Q-N)a	(V-M)a	-(Q-P)a	-(V+Θ)a	-(S+P)a	-(Z-Θ)a	17
	(G+L)	-(F+K)	-(C-J)	-(D-I)	-(E+L)	(H+K)	(A-J)	(B-I)	18
	(W+P)b	(T-Θ)b	-(W-N)b	-(T-M)b	-(U+N)b	-(R+M)b	(U-P)b	(R+Θ)b	19
	-(Q+N)a	-(V+M)a	-(S-N)a	-(Z-M)a	(S-P)a	(Z+Θ)a	(Q+P)a	(V-Θ)a	20

where,

$p = a/b$

$A = 31.5[2p^{-1}+(1-\nu)p]w_{,x}$ $\quad$ $I = 22.5(1+\nu)w_{,x}$ $\quad$ $Q = 4.5(1-\nu)pw_{,x}$

$B = 31.5[2p+(1-\nu)p^{-1}]w_{,y}$ $\quad$ $J = 22.5(1+\nu)w_{,y}$ $\quad$ $R = 4.5(1-\nu)p^{-1}w_{,y}$

$C = 13.5[2p^{-1}+(1-\nu)p]w_{,x}$ $\quad$ $k = 22.5(1-3\nu)w_{,x}$ $\quad$ $S = 3(1-\nu)pw_{,x}$

$D = 13.5[2p+(1-\nu)p^{-1}]w_{,y}$ $\quad$ $L = 22.5(1-3\nu)w_{,y}$ $\quad$ $T = 3(1-\nu)p^{-1}w_{,x}$

$E = [63p^{-1}-13.5(1-\nu)p]w_{,x}$ $\quad$ $M = 3.75(1+\nu)w_{,x}$ $\quad$ $U = 9p^{-1}w_{,x}$

$F = [63p-13.5(1-\nu)p^{-1}]w_{,y}$ $\quad$ $N = 3.75(1+\nu)w_{,y}$ $\quad$ $V = 9pw_{,y}$

$G = [27p^{-1}-31.5(1-\nu)p]w_{,x}$ $\quad$ $\Theta = 3.75(1-3\nu)w_{,x}$ $\quad$ $W = 6p^{-1}w_{,x}$

$H = [27p-31.5(1-\nu)p^{-1}]w_{,y}$ $\quad$ $P = 3.75(1-3\nu)w_{,y}$ $\quad$ $Z = 6pw_{,y}$

The Second-Order Non-linear Element Stiffness Matrix $[K^b_L]$:

$$
[K^b_L] = \frac{Eh}{180(1-\nu^2)}
\begin{array}{c}
\begin{array}{cccccccccccc} 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \end{array} \\
\left[\begin{array}{ccc:ccc:ccc:ccc}
A1 & & & & & & & & & & & \\
-I2 & \theta 2 & & & & & & & & & & \\
I1 & -Z & \theta 1 & & & \text{Symmetric} & & & & & & \\
-R1 & F2 & -J1 & A2 & & & & & & & & \\
F1 & -P2 & Z & -I2 & \theta 2 & & & & & & & \\
J1 & Z & -Q1 & -I1 & -Z & \theta 1 & & & & & & \\
-B1 & D1 & -C1 & R2 & -J2 & -E2 & A1 & & & & & \\
-D1 & H1 & -Z & J2 & -Q2 & Z & I2 & \theta 2 & & & & \\
C1 & -Z & -G2 & -E1 & Z & P1 & -I1 & -Z & \theta 1 & & & \\
R2 & -J2 & E1 & -B2 & D2 & C2 & -R1 & -F2 & J1 & A2 & & \\
J2 & -Q2 & Z & -D2 & H2 & -Z & -F1 & -P2 & Z & I2 & \theta 2 & \\
E2 & Z & P1 & -C2 & -Z & -G1 & -J1 & Z & -Q1 & I1 & -Z & \theta 1
\end{array}\right]
\end{array}
\begin{array}{c} 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \end{array}
$$

$$M=\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}(1-\nu)\left(\frac{\partial w}{\partial y}\right)^2$$

$$N=\left(\frac{\partial w}{\partial y}\right)^2+\frac{1}{2}(1-\nu)\left(\frac{\partial w}{\partial x}\right)^2$$

$$L=\frac{1}{2}(1+\nu)\left(\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right)$$

$$p=a/b$$

where,

$A\genfrac{}{}{0pt}{}{1}{2} = \left(\frac{552}{7}p^{-1}M+\frac{552}{7}pN \pm 90L\right)$

$B\genfrac{}{}{0pt}{}{1}{2} = \left(\frac{204}{7}p^{-1}M+\frac{204}{7}pN \pm 90L\right)$

$C\genfrac{}{}{0pt}{}{1}{2} = \left(3p^{-1}M+\frac{39}{7}pN \pm 18L\right)a$

$D\genfrac{}{}{0pt}{}{1}{2} = \left(\frac{39}{7}p^{-1}M+3pN \pm 18L\right)b$

$E\genfrac{}{}{0pt}{}{1}{2} = \left(3p^{-1}M-\frac{66}{7}pN \pm 18L\right)a$

$F\genfrac{}{}{0pt}{}{1}{2} = \left(\frac{66}{7}p^{-1}M-3pN \pm 18L\right)b$

$G\genfrac{}{}{0pt}{}{1}{2} = \left(p^{-1}M-\frac{9}{7}pN \pm 3L\right)a^2$

$H\genfrac{}{}{0pt}{}{1}{2} = \left(\frac{9}{7}p^{-1}M-pN \pm 3L\right)b^2$

$I1 = \left(6p^{-1}M+\frac{66}{7}pN\right)a$

$I2 = \left(\frac{66}{7}p^{-1}M+6pN\right)b$

$J1 = \left(6p^{-1}M-\frac{39}{7}pN\right)a$

$J2 = \left(\frac{39}{7}p^{-1}M-6pN\right)b$

$\theta 1 = \left(8p^{-1}M+\frac{12}{7}pN\right)a^2$

$\theta 2 = \left(\frac{12}{7}p^{-1}M+8pN\right)b^2$

$P1 = \left(4p^{-1}M-\frac{12}{7}pN\right)a^2$

$P2 = \left(\frac{12}{7}p^{-1}M-4pN\right)b^2$

$Q1 = \left(2p^{-1}M+\frac{9}{7}pN\right)a^2$

$Q2 = \left(\frac{9}{7}p^{-1}M+2pN\right)b^2$

$R1 = \left(\frac{552}{7}p^{-1}M-\frac{204}{7}pN\right)$

$R2 = \left(\frac{204}{7}p^{-1}M-\frac{552}{7}pN\right)$

$Z = 2.5abL$