Alex. Eng. J. Alex. Univ., Volum 28, pp. 185 - 196 (1989)

LOGICAL AND ARITHMETIC AUTOCORRELATION FUNCTIONS

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ABSTRACT

The logical autocorrelation function (LAF) and the arithmetic autocorrelation function (AAF) are studied. A fast algorithm performing the LAF to AAF conversion is presented and explained. A new method is proposed for the computation of the AAF which proves to be more efficient than the conventional ones. Numerical results and comments are also included.

NOTATION

LAF	:	Logical autocorrelation function
AAF	:	Arithmetic autocorrelation function
FFT	:	Fast Fourier transform
FWHT	:	Fast Walsh-Hadamard Transform
x(j)	:	Random sequence of length N .
L(j)	:	LAF sequence
A(j)	:	AAF sequence
E[x]	:	Expectation of x

1. INTRODUCTION

The problem of recovering a known signal immersed in noise has been receiving increased interest in many areas, especially in communication: detection, identification, recognition and matched filtering, [1] -[3]. The correlation function is the basic tool generally used for this purpose.

In the present work, we introduce the logical autocorrelation function (LAF) and the arithmetic autocorrelation function (AAF), (sections 2, 3). The relations and conversions between them are studied (section 4). Different methods for computing the AAF are proposed (section 5).

2. LOGICAL AUTOCORRELATION FUNCTION (LAF)

We consider a wide-sense stationary random process composed of M blocks or "windows", each represented by a random sequence x(j) of length N. The local LAF based on the m^{th} window is defined as, [4]:

$$L^m(j) = 1/N \sum_{i=o}^{N-1} x(i)\, x(i \oplus j) \quad (1)$$

where $(i \oplus j)$ denotes the modulo-2 sum of the integers i and j. The procedure involved in (1) is also known as dyadic correlation.

The logical autocorrelation function (LAF), denoted by L(j), is the expected value of the local LAF's,

$$L(j) = E[L^m(j)] = 1/M \sum_{m=1}^{M} L^m(j) \quad (2)$$

3. ARITHMETIC AUTOCORRELATION FUNCTION (AAF)

For the random process composed of M windows each represented by a random N-sequence x(j), the local AAF, based on the m^{th} window and denoted by $A^m(j)$, is defined as:

$$A^m(j) = 1/N \sum_{i=0}^{N-1} x(i)\, x(i+j) \quad (3)$$

The arithmetic autocorrelation function (AAF), denoted by A(j), is the expected value of the local AAF's,

$$A(j) = E[A^m(j)] = 1/M \sum_{m=1}^{M} A^m(j) \quad (4)$$

Interrelations between the LAF and AAF have been derived in [5]. It is important, however, to have a more efficient way of performing the LAF to AAF transformation.

4. FAST ALGORITHM FOR LAF TO AAF CONVERSION

In this section, we propose a fast algorithm for the LAF to AAF transformation. Consider the N-element LAF vector $L(0), L(1), \ldots, L(N-1)$, $N = 2^r$.

Step 1: Transition from L to L_1

The vector $L(j)$, $j = 0, 1, \ldots, N-1$, is multiplied by the coefficient vector $[1, c_1, c_2, \ldots, c_{n-1}]^T$ to get the vector $L_1(j)$ where

$$c_j = 2^{R_j - 1}$$

R_j is the number of 1's in the binary representation of j. (Superscript T denotes the transpose of the vector).

Step 2: Transition from L_1 to L_2

The elements of the vector $L_1(j)$ are subdivided into $2^{r-1} = N/2$ groups. Elements of the odd numbered groups are unchanged. The first element $L_1(s)$ in an even numbered group is unchanged while the remaining elements of that group are modified according to

$$L_2(s+i) = L_1(s+i) - L_1(s-i)$$

where "i" is a running index within the considered group.

(General) Step k: Transition from L_{k-1} to L_k $(1<k<r)$

The elements of the vector $L_{k-1}(j)$ are subdivided into $2^{r-k+1} = N/2^{k-1}$ groups. Elements of the odd numbered groups are unchanged. The first element $L_{k-1}(s)$ in an even numbered group is unchanged while the remaining elements of that group are modified according to:

$$L_k\,(s+i) = L_{k-1}\,(s+i) - L_{k-1}\,(s-i)$$

i being a running index within the considered group. This is to be repeated for all even numbered groups, corresponding to different values of s.

(Final) Step r: Transition from L_{r-1} to L_r $(r = \mathrm{Log}_2 N)$

Elements of the vector $L_{r-1}(j)$ are subdivided into $2(=2^1)$ groups. Elements of the first group are unchanged.

For the second group:

$$L_r(N/2) = L_{r-1}(N/2)$$
$$L_r(N/2+i) = L_{r-1}(N/2+i) - L_{r-1}(N/2-i) \qquad 1 \leqslant i \leqslant N/2-1$$

The total number of steps required are thus $r = \mathrm{Log}_2 N$. The AAF coefficients are given by:

$$A(i) = L_r(i)\ , \qquad 0 \leqslant i \leqslant N-1$$

The procedure is illustrated in Fig. 1, for the case of N=8.

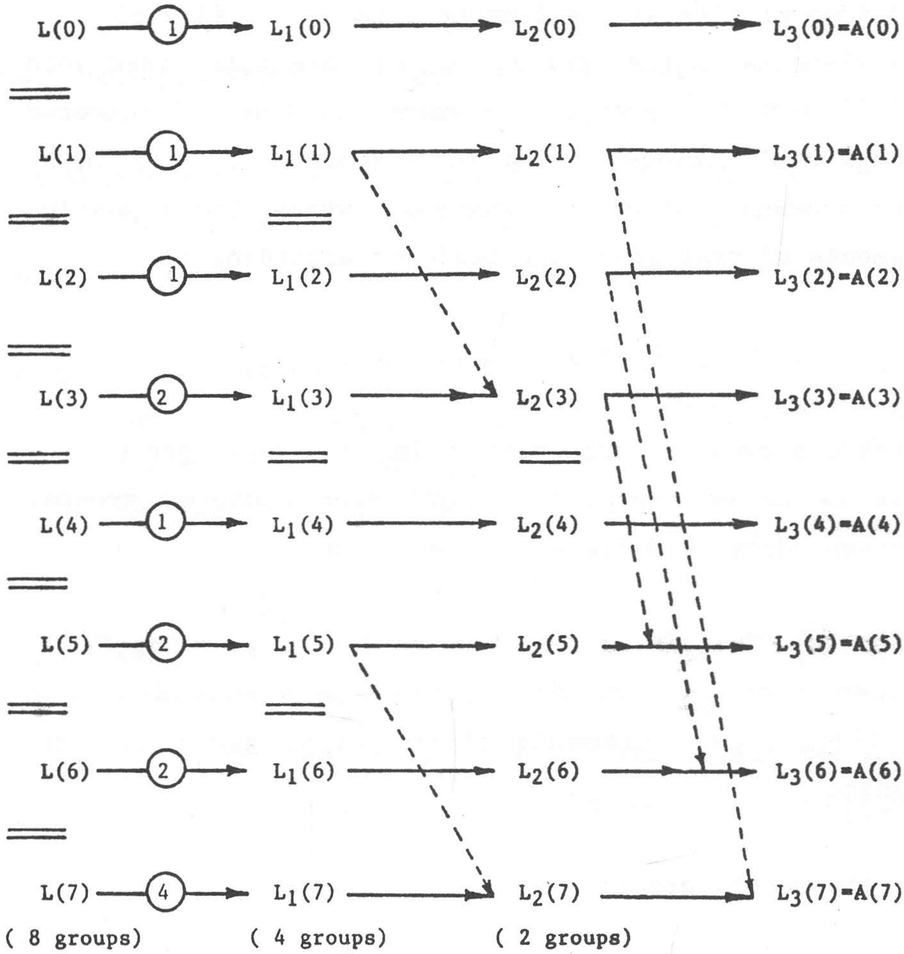
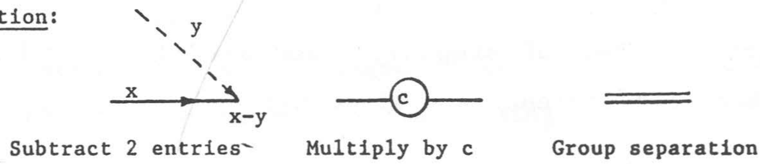


Fig. 1 Fast algorithm for LAF to AAF conversion, N = 8 .

Notation:



5. METHODS FOR THE COMPUTATION OF THE AAF

Two methods are generally used for the computation of the autocorrelation function (AAF), namely the direct method and the FFT method. We introduce a third method based on the very efficient Fast Walsh-Hadamard transformation (FWHT).

5.1 The Direct Method

For a wide-sense stationary random process consisting of M windows (each of N samples), the AAF is computed directly using the local AAF's for the M windows (equations (3),(4)). Each $A^m(j)$ requires N^2 multiplications and N(N-1) additions. For M windows, we have MN^2 multiplications and MN(N-1) additions. The averaging step requires also N(M-1) additions. The total number of operations (S_1) is thus:

$$S_1 = MN^2 + MN(N-1) + N(M-1) = N(2MN-1) \quad (5)$$

5.2 The fast Fourier Transform (FFT) Method

In the FFT method, the AAF is computed as follows:

a. The sequence x(j) of the m^{th} window is Fourier transformed to get the sequence $X_F^m(f)$.

b. The Fourier power spectrum $P_F^m(f)$ of the m^{th} window is obtained by squaring $X_F^m(f)$ at each frequency f,

$$P_F^m(f) = [X_F^m(f)]^2$$

c. The (total) Fourier power spectrum $P_F(f)$ is obtained by averaging $P_F{}^m(f)$ over the M windows,

$$P_F(f) = 1/M \sum_{m=1}^{M} P_F{}^m(f)$$

d. Inverse FFT performed on P_F gives the required AAF (since AAF and P_F constitute a Fourier transform pair).

M - FFT's and one inverse FFT are used which require $(M+1).N\log_2 N$ complex multiplications. Additional MN complex multiplications are required to get the Fourier power spectra and N(M-1) complex additions for the averaging process. The total number of operations is thus given by:

$$S_2 = 2 \quad [(M+1)\ N \log_2 N + MN + N(M-1)] \qquad (6)$$

5.3 The Fast Walsh-Hadamard Transform (FWHT) method

The method we propose is based on the use of the fast Walsh-Hadamard transform (FWHT) together with the fast LAF to AAF conversion algorithm presented in section 4. It thus combines their inherent advantages as regard ease and efficiency of computation.

The main steps in the procedure are as follows:

a. The sequence x(j) of the m^{th} window is Walsh transformed, using the FWHT, to get the sequence $X_w^m(k)$. (Efficient FWHT subroutines are given in [5]).

b. The Walsh power spectrum $P_w^m(k)$ of the m^{th} window is obtained by squaring $X_w^m(k)$ at each sequency k,

$$P_w^m(k) = [X_w^m(k)]^2$$

c. The previous steps are carried out for all windows. The (total) Walsh power spectrum $P_w(k)$ is then obtained by averaging $P_w^m(k)$ over the M windows.

d. Inverse FWHT performed on $P_w(k)$ gives the LAF (since LAF and $P_w(k)$ constitute a Walsh transform pair).

e. Use of the LAF to AAF fast transformation routine (section 4), yields the required AAF . The total number of operations in this case is given by,

$$S_3 = (M+1)N\log_2 N + MN + N(M-1) + (N/2)\log_2 N - \log_2 N + N/2 + 1$$

$$S_3 = (MN + 3N/2 - 1)\log_2 N + N(2M-1/2) + 1 \qquad (7)$$

Table I gives the total number S of operations required by each of the three methods for the case of M= 8 windows and for different number of samples N.

It is evident from Table I that the proposed FWHT method requires a number of operations much less than that of the FFT method or the direct method.

Table I: Number of operations for AAF computation

N	Direct method S_1	FFT method S_2	Proposed FWHT Method S_3
4	252	260	137
8	1016	672	350
16	4080	1632	853
32	16352	3840	2012
64	65472	8832	4635

6. NUMERICAL RESULTS

We have used the different methods of section 5 to compute the LAF's and the AAF's of many useful signals. Some illustrative results are shown in Figs. 2 to 5 for the case of rectangular, triangular, decreasing exponential and cosine signals.

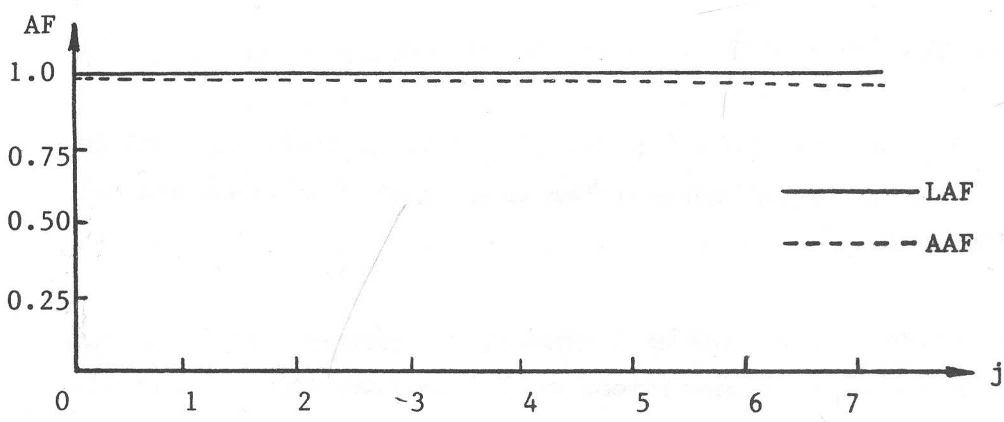


Fig. 2 The LAF and the AAF of a rectangular signal .

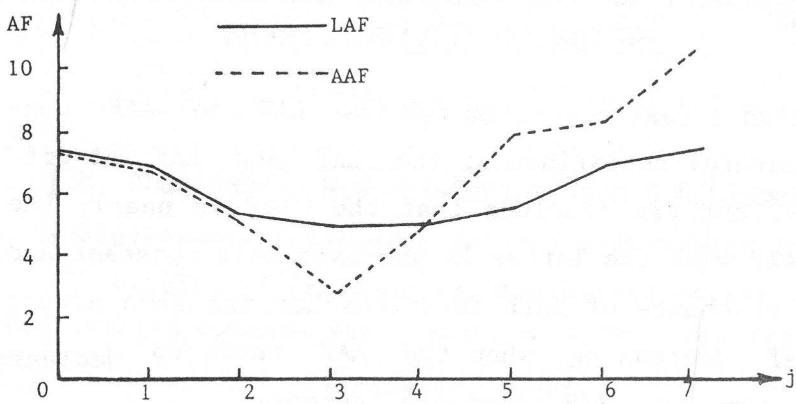


Fig. 3 The LAF and the AAF of a triangular signal .

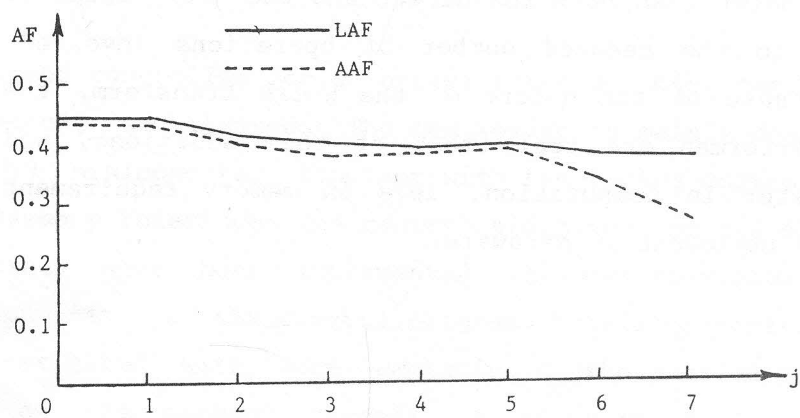


Fig. 4 The LAF and the AAF of a decreasing exponential signal.

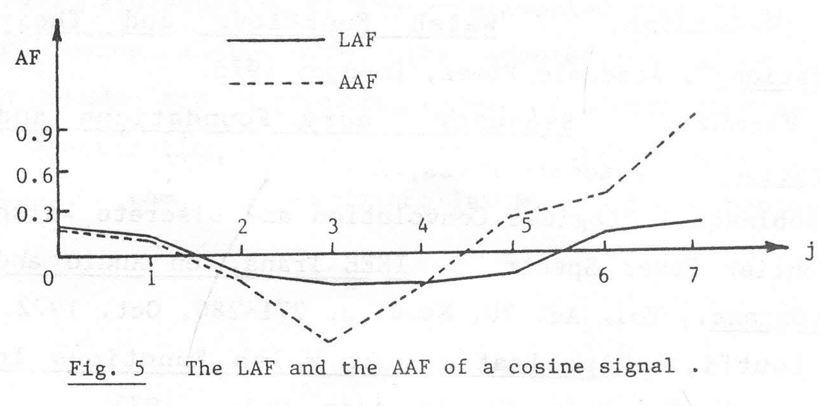


Fig. 5 The LAF and the AAF of a cosine signal .

7. CONCLUSIONS

We have presented a fast algorithm for the LAF to AAF conversion. By careful comparison of the LAF and AAF of different signals, one can conclude that the LAF is nearly the same as the AAF when the latter is approximately constant and that the rate of change of both functions has the same direction, (the LAF decreasing when the AAF tends to decrease and increasing when the AAF tends to increase).
The FWHT method we propose computes the AAF appreciably faster and simpler than both the direct and the FFT methods. This is due to the reduced number of operations involved. Moreover, because of the nature of the Walsh transform, the operations performed are real (most of them additions) and therefore faster in computation, less in memory requirement and easier to implement in hardware.

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