Alex. Eng. J., Alex. Univ., Volume 28, pp. 95 - 120 (1989)

A STATE SPACE REPRESENTATION OF SHIP STEERING DYNAMICS

Aly M. El-Iraki and M. Hanafi Soliman

Marine Engineering and Naval Architecture Department

Faculty of Engineering, Alexandria University

Alexandria, Egypt

Abstract

A state space model was developed for ship steering dynamics. The transfer matrix, state and output matrix equations for drift angle and yaw rate for a general function of rudder deflection were derived as well as the state variable diagram and state variable signal flow graph. The solution for the drift angle and yaw rate was explicity given for a step input.

Nomenclature

1. Coordinates:

$x_o y_o z_o$ inertial coordinate system

xyz hull-fixed coordinate system with origin at ⊗

x_{oG} x-coordinate of ship's center of gravity in $x_o y_o z_o$- system

y_{oG} y-coordinate of ship's center of gravity in $x_o y_o z_o$- system

x_G x-coordinate of ship's center of gravity in xyz-system

2. Forces and moments

X_o hydrodynamic force acting on ship in x_o-direction

X hydrodynamic force acting on ship in x-direction

Y_o hydrodynamic force acting on ship in y_o-direction

Y hydrodynamic force acting on ship in y-direction

N_G hydrodynamic moment acting on ship about a vertical axis through G

N hydrodynamic moment acting on ship about z - axis

3. Ship particulars:

L ship's length

d ship's draft

m ship's mass

I_{zz} mass moment of inertia of ship about a vertical axis through G

V ship's resultant velocity

4. Steering variables

β drift angle

ψ yaw angle

δ rudder angle

u component of ship's velocity V in x-direction

v component of ship's velocity V in y-direction

$y_1 = \beta_1\,(t')$, $y_2 = r'(t')$ states of the dynamic system

5. Nondimensional quantities

$u' = u/V$

$v' = v/V$

$m' = m/(\frac{1}{2}\rho L^2 d)$

$m'_y = m' - Y'_{\dot{v}}$

$I'_{zz} = I_{zz}/(\frac{1}{2}\rho L^4 d)$

$i'_z = I'_{zz} - N'_{\dot{r}}$

$X'_G = X_G/L$

$r' = r\,L/V$

$t' = t\,V/L$

6. Hydrodynamic derivatives and their dimensionless form:

$X_u = \partial X/\partial u$

$X_{\dot{u}} = \partial X/\partial \dot{u}$

$X_\delta = \partial X/\partial \delta$

$Y_v = \partial Y/\partial v$ $\quad Y'_v = Y_v/(\frac{1}{2}\rho LdV) = -Y'_\beta$

$Y_r = \partial Y/\partial r$ $\quad Y'_r = Y_r/(\frac{1}{2}\rho L^2 dV)$

$Y_{\dot{v}} = \partial Y/\partial \dot{v}$ $\quad Y'_{\dot{v}} = Y_{\dot{v}}/(\frac{1}{2}\rho L^2 d) = -Y'_{\dot{\beta}}$

$Y_{\dot{r}} = \partial Y/\partial \dot{r}$ $\quad Y'_{\dot{r}} = Y_{\dot{r}}/(\frac{1}{2}\rho L^3 d)$

$Y_\delta = \partial Y/\partial \delta$ $\quad Y'_\delta = Y_\delta/(\frac{1}{2}\rho LdV^2)$

$N_v = \partial N/\partial v$ $\quad N'_v = N_v/(\frac{1}{2}\rho L^2 dV) = -N'_\beta$

$N_r = \partial N/\partial r$ $\quad N'_r = N_r/(\frac{1}{2}\rho L^3 dV)$

$N_{\dot{v}} = \partial N / \partial \dot{v}$ $\qquad N'_{\dot{v}} = N_{\dot{v}} / (\frac{1}{2} \rho L^2 d V) = - N'_{\dot{\beta}}$

$N_{\dot{r}} = \partial N / \partial \dot{r}$ $\qquad N'_{\dot{r}} = N_{\dot{r}} / (\frac{1}{2} \rho L^4 d)$

$N_{\delta} = \partial N / \partial \delta$ $\qquad N'_{\delta} = N_{\delta} / (\frac{1}{2} \rho L^2 d V^2)$

7. Miscellaneous

t time

ρ density of water

$(\ )^{\bullet}$ differentiation with respect to t for dimensional quantities, and with respect to t' for dimensionless quantities

1. Introduction

Course stability and steering of ships are among the main concerns of the ship designer. These problems lie in essence in the domain of dynamics. Denis and Craven [1] classified them indeed as control problems. The complexity of such problems, however, arises from the interaction between the ship and the surrounding medium, the water.

The methods of solution of such problems are very closely related to those adopted in dynamics and control systems treatment. Nevertheless, modern solution techniques, such as the state space technique, were obviously not applied to ship motions problems. The transition to such technique is however advantageous. The state space approach, which is a time domain one, allows the inclusion of the initial conditions of the system, a task which may prove to be difficult when using conventional techniques. Moreover, complex dynamic systems such as multi-variable, nonlinear, time varying, stochastic or sampled-data systems can be treated using the state space approach for analysis, design and synthesis tasks. This technique also renders itself readily to digital computation as it adopts matrix representation. The decomposition of the block diagram of the system into state variable diagram allows the use of analog simulation.

The aim of this paper is to apply the state space method to the problem of plane ship motions in calm water and in the absence of wind effects.

2. Equations of Motion

Of the six possible rigid ship motions, the plane ones, namely surge, sway and yaw, are closely connected to the problem of stability and manoeuvring. They can be treated, especially for large ships, independent of the heave, roll and pitch motions [2].

In an inertial coordinate system $x_o y_o z_o$, see Fig. (1), the equations of motion of a ship can be written as [3,4]:

$$\begin{aligned} m\ddot{x}_{oG} &= X_o \quad \text{(surge)} \\ m\ddot{y}_{oG} &= Y_o \quad \text{(sway)} \\ I_{zz}\,\dot{r} &= N_G \quad \text{(yaw)} \end{aligned} \qquad (1)$$

In spite of the simplicity of the equations of motions in an inertial system, it is more convenient to formulate them in a hull-fixed coordinate system. This is, on one hand, because the hydrodynamic forces and moments acting on a ship depend on her motion and orientation relative to the surrounding environment, and hence it is easier to express them in hull-fixed coordinates. On the other hand, these hydrodynamic forces and moments are normally determined experimentally through model tests. The measuring devices in such tests usually travel with the model, i.e. the forces and moments are measured and related to a hull-fixed coordinate system, and the

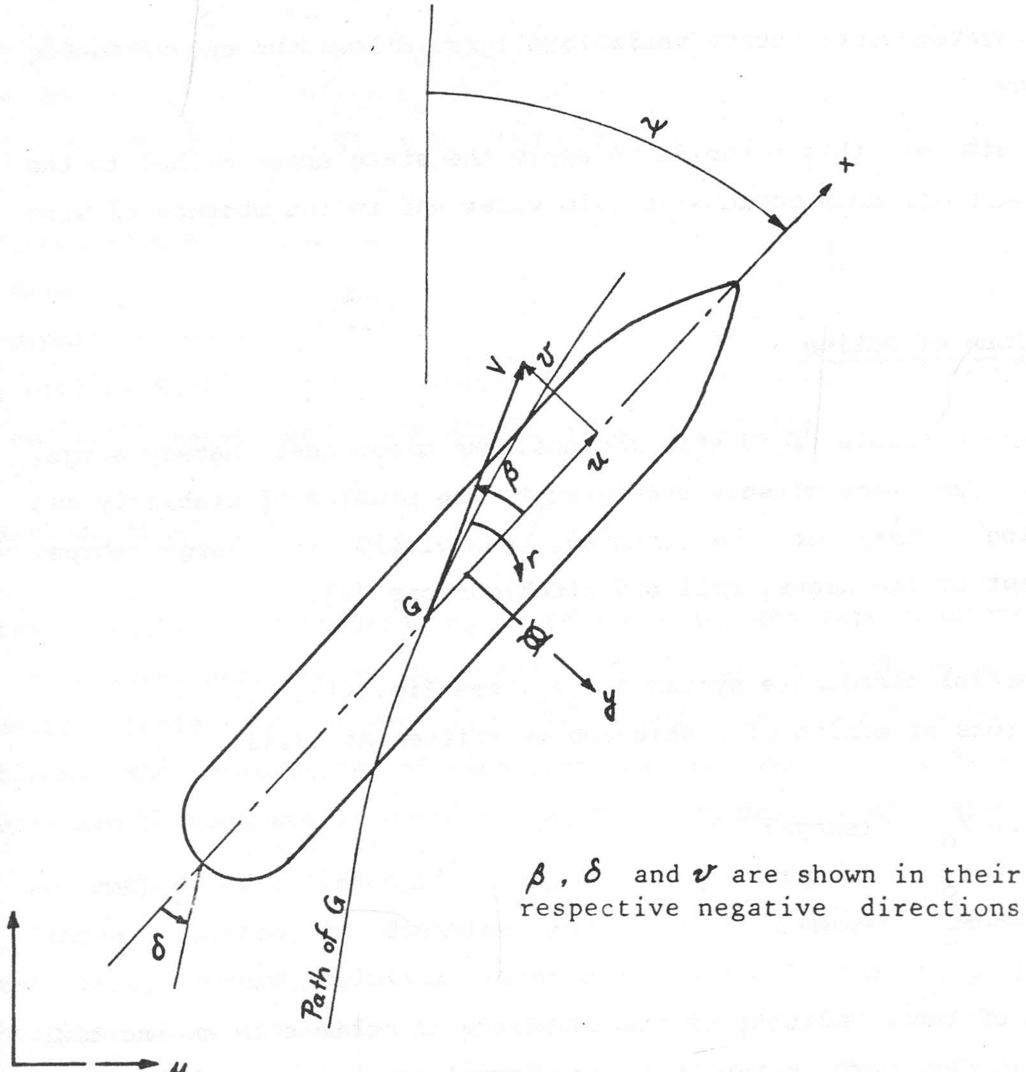

Fig. 1 Coordinates to Describe Ship Motions

corresponding mathematical model is conveniently expressed in such coordinates.

Two distinct points are considered for defining the origin of the hull-fixed coordinate system. The first is, naturally, the center of gravity of the ship (usually assumed to lie in the center plane of symmetry of the ship). In this case, the equations of motion take the form [3,4,5,6]

$$\begin{aligned} m(\dot{u} - vr) &= X \\ m(\dot{v} - ur) &= Y \\ I_{zz}\,\dot{r} &= N_G \end{aligned} \tag{2}$$

The center of gravity of a ship is, however, not a fixed point since its position varies with the loading of the ship. It is mainly because of this fact that the origin of the hull-fixed coordinate system is preferably located amidships. The equations of motion in this case take the form [2,3,5].

$$\begin{aligned} m(\dot{u} - vr - x_G\, r^2) &= X \\ m(\dot{v} + ur + x_G\, \dot{r}) &= Y \\ (I_{zz} + mX_G^2)\dot{r} + mX_G(\dot{v}+ur) &= N = N_G + Y.X_G \end{aligned} \tag{3}$$

The hydrodynamic forces X and Y and the hydrodynamic moment N are considered to be linear functions of the motion of the ship and the rudder and the derivatives thereof, i.e.

$$\begin{aligned} X &= X\,(u, v, r, \dot{u}, \dot{v}, \dot{r}, \delta) \\ Y &= Y\,(u, v, r, \dot{u}, \dot{v}, \dot{r}, \delta) \\ N &= N\,(u, v, r, \dot{u}, \dot{v}, \dot{r}, \delta) \end{aligned} \tag{4}$$

Using linear Taylor expansion, the functions in Eqns. (4) are expanded about the initial ship's equilibrium position given by

$$u = u_o$$
$$\dot{u}_o = v_o = \dot{v}_o = r_o = \dot{r}_o = \delta_o = 0, \qquad (5)$$

where the suffix "O" denotes the equilibrium position corresponding to a ship moving on a straight course at constant speed with zero rudder angle.

Hence,

$$X = \frac{\partial X}{\partial u}(u - u_o) + \frac{\partial X}{\partial v} v + \frac{\partial X}{\partial r} r + \frac{\partial X}{\partial \dot{u}} \dot{u} + \frac{\partial X}{\partial \dot{v}} \dot{v} + \frac{\partial X}{\partial \dot{r}} \dot{r} + \frac{\partial X}{\partial \delta}\delta$$

$$Y = \frac{\partial Y}{\partial u}(u - u_o) + \frac{\partial Y}{\partial v} v + \frac{\partial Y}{\partial r} r + \frac{\partial Y}{\partial \dot{u}} \dot{u} + \frac{\partial Y}{\partial \dot{v}} \dot{v} + \frac{\partial Y}{\partial \dot{r}} \dot{r} + \frac{\partial Y}{\partial \delta}\delta \qquad (6)$$

$$N = \frac{\partial N}{\partial u}(u - u_o) + \frac{\partial N}{\partial v} v + \frac{\partial N}{\partial r} r + \frac{\partial N}{\partial \dot{u}} \dot{u} + \frac{\partial N}{\partial \dot{v}} \dot{v} + \frac{\partial N}{\partial \dot{r}} \dot{r} + \frac{\partial N}{\partial \delta}\delta$$

The coefficients on the right hand side of Eqns. (6) are normally called the derivatives of the hydrodynamic forces and moments, or simply the hydrodynamic derivatives.

Because of the symmetry of the ship relative to here longitudinal center plane, some of these hydrodynamic derivatives vanish [3]. Eqns. (6) simplify then to

$$X = X_u (u - u_o) + X_{\dot{u}} \dot{u} + X_\delta \delta$$
$$Y = Y_v v + Y_r r + Y_{\dot{v}} \dot{v} + Y_{\dot{r}} \dot{r} - Y_\delta \delta \qquad (7)$$
$$N = N_v v + N_r r + N_{\dot{v}} \dot{v} + N_{\dot{r}} \dot{r} + N_\delta \delta$$

Since the hydrodynamic forces are now expressed in linear form, it is required to linearize also the left hand side of Eqns. (3) as they include the nonlinear terms vr, ur and r^2.

After carrying out this linearization,
Eqns. (3) can now be written as

$$
\begin{aligned}
m\dot{u} &= X_u (u - u_o) + X_{\dot{u}} \dot{u} + X_{\delta}\delta \\
m(\dot{v}+u_o r + X_G \dot{r}) &= Y_v v + Y_r r + Y_{\dot{v}} \dot{v} + Y_{\dot{r}} \dot{r} + Y_{\delta}\delta \qquad (8) \\
(I_{zz}+mx_G^2)\, \dot{r} + mX_G(\dot{v}+u_o r) &= N_v v + N_r r + N_{\dot{v}}\dot{v}+N_{\dot{r}}\dot{r}+N_{\delta}\delta
\end{aligned}
$$

or in matrix form with $u_o = V$ [2]

$$
\begin{bmatrix} (m - X_{\dot{u}}) & 0 & 0 \\ 0 & (m-Y_{\dot{v}}) & (mx_G - Y_{\dot{r}}) \\ 0 & (mx_G-N_{\dot{v}}) & (I_{zz}+mx_G^2-N_{\dot{r}}) \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \end{bmatrix} +
$$

$$
+ \begin{bmatrix} - X_u & 0 & 0 \\ 0 & - Y_v & (mV - Y_r) \\ 0 & - N_v & (mx_G V-N_r) \end{bmatrix} \begin{bmatrix} u-v \\ v \\ r \end{bmatrix} = \begin{bmatrix} X_\delta \\ Y_\delta \\ N_\delta \end{bmatrix} \delta \qquad (9)
$$

It is an established fact that linearization of nonlinear control systems is valid only for small deviations near the steady state condition. In our case this corresponds to relatively small rudder angles for directional control. A result of the linearization adopted here is the uncoupling of the surge equation from the sway and yaw equations, as can readily be seen from Eqn. (9). Hence, sway and yaw motions can be treated independent of the surge motion. The coupled sway and yaw equations are

$$\begin{bmatrix} (m-Y_{\dot{v}}) & (mx_G-Y_{\dot{r}}) \\ (mX_G-N_{\dot{v}}) & (I_{zz}+mx_G^2-N_{\dot{r}}) \end{bmatrix} \cdot \begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} -Y_v & (mV-Y_r) \\ -N_v & (mx_G V-N_r) \end{bmatrix} \cdot \begin{bmatrix} v \\ r \end{bmatrix} = \begin{bmatrix} Y_\delta \\ N_\delta \end{bmatrix} \delta \qquad (10)$$

These equations can be written in a dimensionless form as:

$$\begin{bmatrix} (m'-Y'_{\dot{v}}) & (m'x'_G-Y'_{\dot{r}}) \\ (m'X'_G-N'_{\dot{v}}) & (I'_{zz}+m'x'^2_G-N'_{\dot{r}}) \end{bmatrix} \cdot \begin{bmatrix} \dot{v}' \\ \dot{r}' \end{bmatrix} + \begin{bmatrix} -Y'_v & (m'-Y'_r) \\ -N'_v & (m'X'_G - N'_r) \end{bmatrix} \cdot \begin{bmatrix} v' \\ r' \end{bmatrix} = \begin{bmatrix} Y'_\delta \\ N'_\delta \end{bmatrix} \delta \qquad (11)$$

The term $(m'-Y'_{\dot{v}})$ represents the dimensionless virtual mass of the ship in sway, m'_y. It is approximately equal to 2 m', since $Y_{\dot{v}} \approx - m'$ [3].

Similarly, the term $(I'_{zz} - N'_{\dot{r}})$ represents the dimensionless virtaul mass moment of inertia of the ship about her C.G., i'_z. Again, this term approximately equals $2I'_{zz}$ [3].

Utilizing the relation between the drift angle β and the drift velocity v, see Fig. (1),

$$\beta = -\sin^{-1}\frac{v}{V} = -\sin^{-1} v' \approx - v' ,$$

Eqn. (11) can now be expressed in terms of the drift angle β and the diemsnionless yaw rate r'. This yields

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \dot{\beta} \\ \dot{r}' \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} \beta \\ r' \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \cdot \delta , \qquad (12)$$

where

$$
\begin{aligned}
\alpha_{11} &= - m' - Y'_{\dot{\beta}} \\
\alpha_{12} &= m'x'_G - Y'_{\dot{r}} \\
\alpha_{21} &= -m'x'_G - N'_{\dot{\beta}} \\
\alpha_{22} &= i'_z + m'x'^2_G \\
\beta_{11} &= - Y'_{\beta} \\
\beta_{12} &= m' - Y'_r \\
\beta_{21} &= N'_{\beta} \\
\beta_{22} &= m'x'_G - N'_r \\
\gamma_1 &= Y'_{\delta} \\
\gamma_2 &= N'_{\delta}
\end{aligned}
\qquad (13)
$$

3. Mathematical Modelling

Taking Laplace transform of Eqn. (12) yields

$$
\begin{bmatrix} \alpha_{11}S + \beta_{11} & \alpha_{12}S + \beta_{12} \\ \alpha_{21}S + \beta_{21} & \alpha_{22}S + \beta_{22} \end{bmatrix} \begin{bmatrix} \beta(s) \\ r'(s) \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \cdot \delta(S) \qquad (14)
$$

Fig. (2a) is the block diagram representation of Eqn. (14). The output vector in terms of the transfer matrix and the input rudder angle δ is

$$
\begin{bmatrix} \beta(S) \\ r'(s) \end{bmatrix} = \begin{bmatrix} \dfrac{\gamma_1(\alpha_{22}S + \beta_{22}) - \gamma_2(\alpha_{12}S + \beta_{12})}{\Delta} \\ \\ \dfrac{-\gamma_1(\alpha_{21}s + \beta_{21}) + \gamma_2(\alpha_{11}S + \beta_{11})}{\Delta} \end{bmatrix} \cdot \delta(S) \qquad (15)
$$

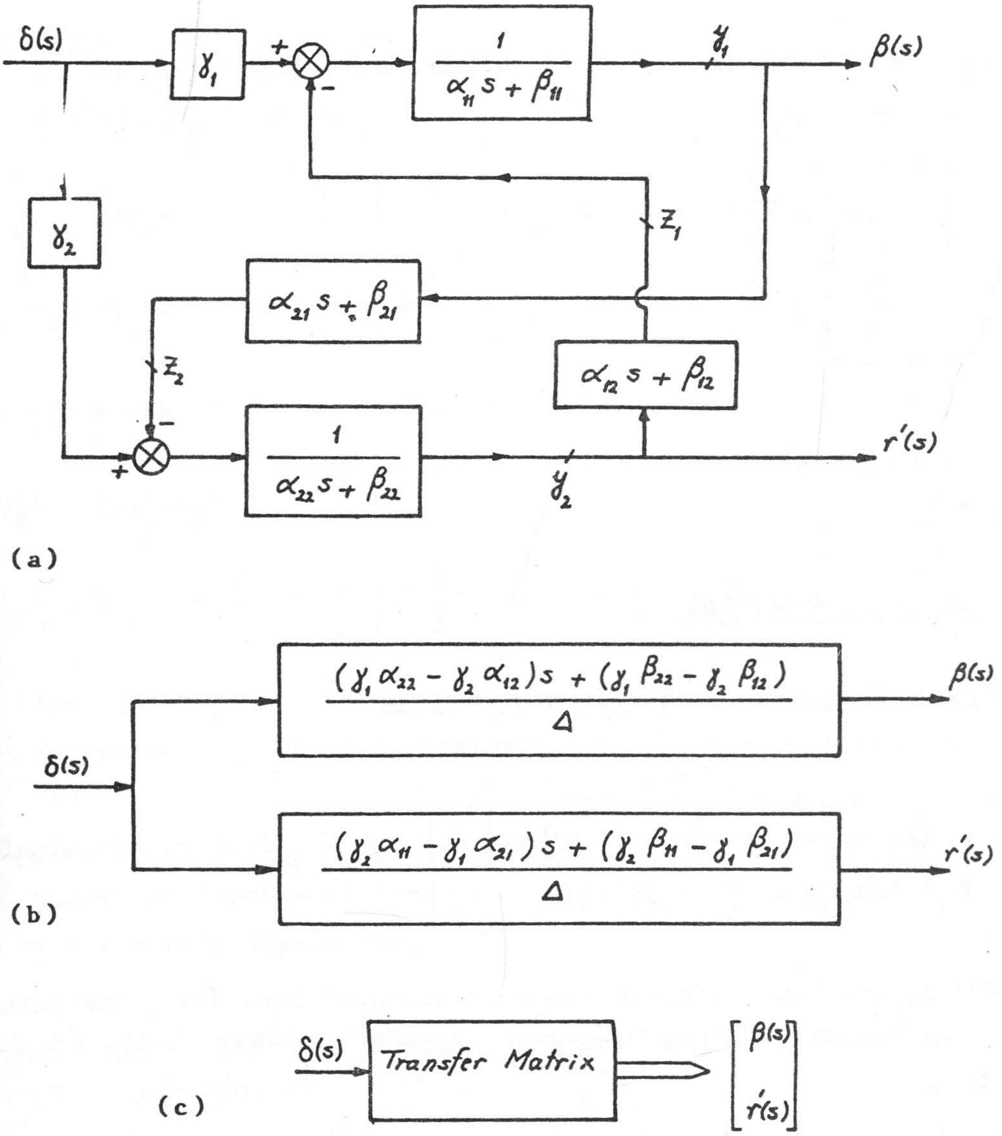

Fig.2 Alternative forms of the block diagram of ship steering dynamics

where

$$\Delta = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})S^2 + (\beta_{11}\alpha_{22} + \alpha_{11}\beta_{22} - \alpha_{21}\beta_{12} - \alpha_{12}\beta_{21}) +$$

$$+ (\beta_{11}\beta_{22} - \beta_{12}\beta_{21}) \neq 0$$

All the roots of the characteristic equation $\Delta = 0$ should have negative real parts to ensure the stability of the dynamic system.

Eqn. (15) is represented in block diagram form in Fig. (2b,c).

4. State Space Representation

The state of a dynamic system is defined as the smallest set of variables, the state variables, such that the knowledge of these variables at time $t = t_o$ together with the input for $t \geq t_o$ completely determines the behavior of the system for any time $t \geq t_o$. The decomposition of the block diagram or transfer function (or matrix) into state and output matrix equations is not unique, while recomposition of the transfer function (or matrix) from the state and output matrix equations is unique.

For the present problem the number of state variables is two since the characteristic equation is of the second degree. The state variables are selected to be the drift angle and the yaw rate denoted by $y_1 = \beta(t')$ and $y_2 = r'(t')$, respectively. Z_1 and Z_2 in Fig. (2a) denote intermediate parameters.

From Fig. (2a) the following differential relations may be deduced according to the method given in [7]

$$\dot{y}_1 = \frac{1}{\alpha_{11}} (\delta \gamma_1 - Z_1 - \beta_{11} y_1) \quad (16)$$

$$\dot{y}_2 = \frac{1}{\alpha_{22}} (\delta \gamma_2 - Z_2 - \beta_{22} y_2) \quad (17)$$

$$Z_1 = \alpha_{12} \dot{y}_2 + \beta_{12} y_2 \quad (18)$$

$$Z_2 = \alpha_{21} \dot{y}_1 + \beta_{21} y_1 \quad (19)$$

Substituting Z_1 and Z_2 from Eqns. (18) and (19) into Eqns. (16) and (17) we obtain the state matrix equation as follows:

$$\begin{bmatrix} \dot{y}_1(t') \\ \dot{y}_2(t') \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \cdot \begin{bmatrix} y_1(t') \\ y_2(t') \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \cdot \delta(t') \quad (20)$$

which could be put in the form

$$[\dot{y}(t')] = [P][y(t')] + [B] \delta(t').$$

where

$$P_{11} = \frac{\alpha_{12} \beta_{21} - \alpha_{22} \beta_{11}}{\alpha_{11} \alpha_{22} - \alpha_{21} \alpha_{12}}$$

$$P_{12} = \frac{\alpha_{12} \beta_{22} - \alpha_{22} \beta_{12}}{\alpha_{11} \alpha_{22} - \alpha_{21} \alpha_{12}}$$

$$P_{21} = \frac{\alpha_{21}\ \beta_{11} - \alpha_{11}\ \beta_{21}}{\alpha_{11}\ \alpha_{22} - \alpha_{21}\ \alpha_{12}}$$

$$P_{22} = \frac{\alpha_{21}\ \beta_{12} - \alpha_{11}\ \beta_{22}}{\alpha_{11}\ \alpha_{22} - \alpha_{21}\ \alpha_{12}}$$

$$B_{1} = \frac{\alpha_{22}\ \gamma_{1} - \alpha_{12}\ \gamma_{2}}{\alpha_{11}\ \alpha_{22} - \alpha_{21}\ \alpha_{12}}$$

$$B_{2} = \frac{\alpha_{11}\ \gamma_{2} - \alpha_{21}\ \gamma_{1}}{\alpha_{11}\ \alpha_{22} - \alpha_{21}\ \alpha_{12}}$$

The output matrix equation is

$$\begin{bmatrix} \beta\ (t') \\ r'(t') \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1(t') \\ y_2(t') \end{bmatrix} \qquad (21)$$

which could be written in the form

$$[c(t')] = [L]\ [y(t')]\ .$$

Equations (20) and (21) are represented graphically in state variable diagram and state variable signal flow graph in Figs. (3) and (4), respectively. The state variable diagram, Fig. (3), is a decomposed form of the block diagram shown in Fig. (2a).

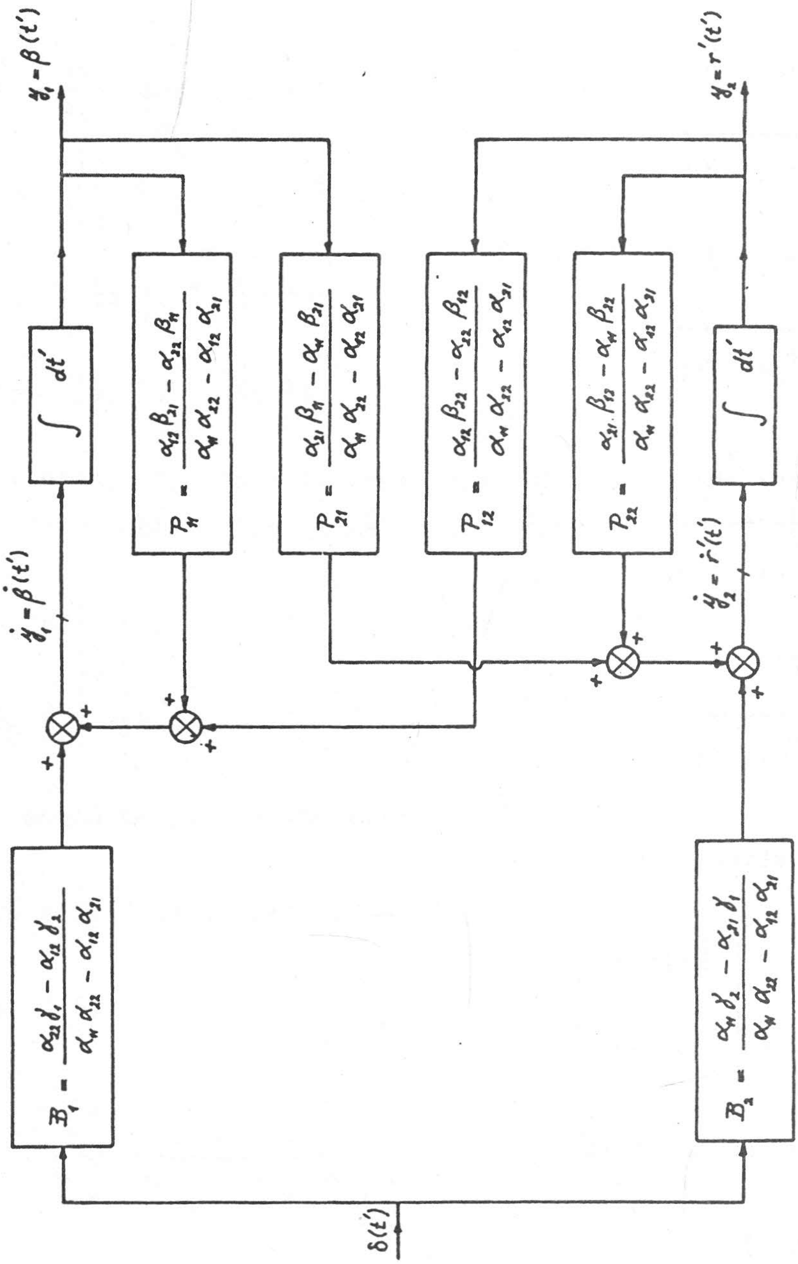

Fig. 3 State Variable Diagram

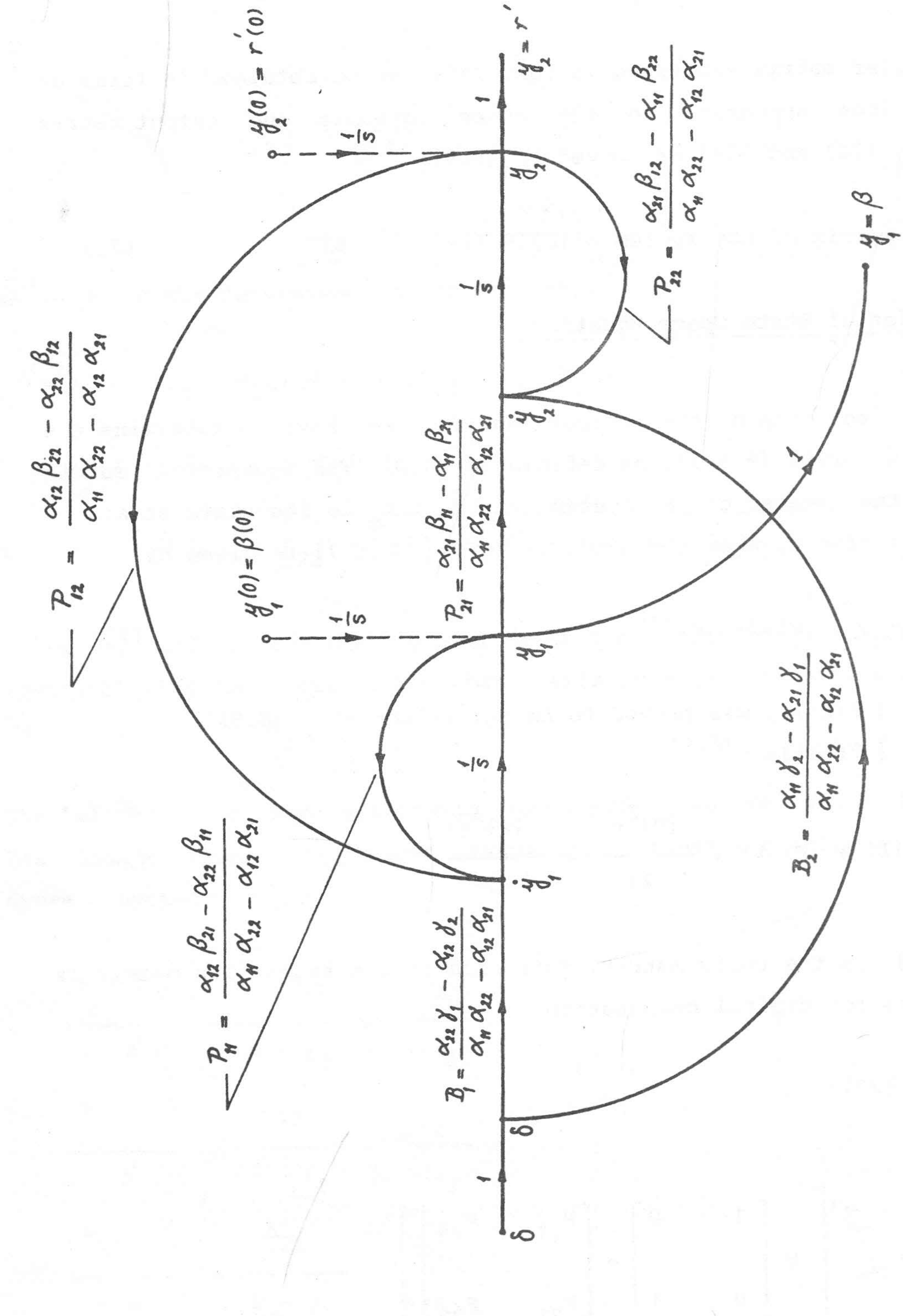

Fig. 4 State Variable Signal Flow Graph

The transfer matrix expressed in Eqn. (15) can be obtained in terms of the matrices appearing in the state variable and output matrix equations (20) and (21) as proved in [8,9]:

Transfer matrix of the system = $[L][S[I]-[P]]^{-1}[B]$ (22)

5. <u>Solution of State Space Equations</u>

In order to obtain the output matrix, we have to determine the transition matrix $[\phi(t')]$, as defined in [8,9]. The transition matrix relates the state of a system at $t = t_o$ to its state at some subsequent time t, when the input is zero. $[\phi(t')]$ is given by

$$[\phi(t')] = \mathcal{L}^{-1}[s[I]-[P]]^{-1} \qquad (23)$$

Moreover, $[\phi(t')]$, was proved to be equivalent to [8,9]

$$[\phi(t')] = e^{[P]t'}$$

$$[\phi(t')] = [I] + [P]t' + \frac{[P]^2 t'^2}{2!} + \frac{[P]^3 t'^3}{3!} + \ldots \qquad (24)$$

where [I] is the unity matrix. This form of the transition matrix is appropriate for digital computation.

From Eqn. (23)

$$[\phi(t')] = \mathcal{L}^{-1}\left[s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \right]^{-1}$$

or

$$[\phi(t')] = \mathcal{L}^{-1} \begin{bmatrix} \frac{s - P_{22}}{\Delta'} & \frac{P_{12}}{\Delta'} \\ \frac{P_{21}}{\Delta'} & \frac{s-P_{11}}{\Delta'} \end{bmatrix} , \qquad (25)$$

where Δ' is the determinant of $[s[I] - [P]]^{-1}$

$$\Delta' = s^2 - (P_{11} + P_{22}) s + (P_{11}P_{22} - P_{12} P_{21})$$

$$= (S + \chi_1)(s + \chi_2) \neq 0$$

$$\chi_{1,2} = \tfrac{1}{2} [- (P_{11} + P_{22}) \pm \sqrt{(P_{11}+P_{22})^2 - 4(P_{11}P_{22}-P_{12}P_{21})}] ,$$

where χ_1 and χ_2 are the eigenvalues of the matrix [P] which are identical with the roots of the characteristic equation $\Delta = 0$, since $\Delta'/\Delta=$ const.

The elements of the matrix in Eqn. (25) can be given by the Heaviside's expansion theorem assuming distinct roots for a stable dynamic system

$$\frac{S - P_{22}}{\Delta'} = \frac{A_{11}}{s + \chi_1} + \frac{C_{11}}{s+ \chi_2}$$

$$\frac{P_{12}}{\Delta'} = \frac{A_{12}}{s + \chi_1} + \frac{C_{12}}{s+\chi_2}$$

$$\frac{P_{21}}{\Delta'} = \frac{A_{21}}{s + \chi_1} + \frac{C_{21}}{s+\chi_2}$$

$$\frac{S - P_{11}}{\Delta'} = \frac{A_{22}}{s + X_1} + \frac{C_{22}}{s+X_2}$$

where

$$A_{11} = \lim_{S \to -X_1} \frac{(s+ X_1)(s-P_{22})}{\Delta'} = \frac{- X_1 - P_{22}}{X_2 - X_1}$$

and in a similar mannar

$$A_{12} = \frac{P_{12}}{X_2 - X_1}$$

$$A_{21} = \frac{P_{21}}{X_2 - X_1}$$

$$A_{22} = \frac{-X_1 - P_{11}}{X_2 - X_1}$$

$$C_{11} = \frac{X_2 + P_{22}}{X_2 - X_1}$$

$$C_{12} = \frac{- P_{12}}{X_2 - X_1}$$

$$C_{21} = \frac{-P_{21}}{\chi_2 - \chi_1}$$

$$C_{22} = \frac{\chi_2 + P_{11}}{\chi_2 - \chi_1}$$

Hence,

$$[\phi(t')] = \begin{bmatrix} \{A_{11}e^{-\chi_1 t'} + C_{11}e^{-\chi_2 t'}\} & \{A_{12}e^{-\chi_1 t'} + C_{12}e^{-\chi_2 t'}\} \\ \{A_{21}e^{-\chi_1 t'} + C_{21}e^{-\chi_2 t'}\} & \{A_{22}e^{-\chi_1 t'} + C_{22}e^{-\chi_2 t'}\} \end{bmatrix}$$

The state vector in time domain could be written as [9]

$$[y(t')] = [\phi(t')][y(0)] + \int_{\tau=0}^{\tau=t'} [\phi(t'-\tau)] [B] \; \delta(\tau) \; d\tau \quad , \tag{26}$$

where τ is the dummy parameter of the convolution integral.
Since the state vector $[y(t')]$ represents the output vector $[c]$, it is obvious that $[L]$ is the unity matrix of order two.
The output vector is

$$[c(t')] = \begin{bmatrix} \beta(t') \\ r'(t') \end{bmatrix} = [L][y(t')] \tag{27}$$

Finally, from Eqns. (26) and (27) the output vector is given by

$$\begin{bmatrix} \beta(t') \\ \\ r'(t') \end{bmatrix} = [L][\phi(t')][y(0)] + \int_{\tau=0}^{\tau=t'} [L][\phi(t'-\tau)][B]\delta(\tau)\, d\tau . \qquad (28)$$

Knowing the input rudder angle $\delta(t')$ and the initial state vector

$$[y(0)] = \begin{bmatrix} y_1(0) \\ \\ y_2(0) \end{bmatrix} = \begin{bmatrix} \beta(0) \\ \\ r'(0) \end{bmatrix}$$

The drift angle $\beta(t')$ and the nondimensional yaw rate $r'(t')$ can be evaluated using Eqn. (28).

Normally the initial state of the ship traveling in straight course before commencing a manoeuvre corresponds to a zero state vector; i.e. no drift angle and no yaw rate.

The input rudder angle $\delta(t')$ corresponds to the type of manoeuvre which the ship is to execute. This may be represented by a pulse input for a change of direction, a unit step for a turn or a trapizoidal wave for the Z-manoeuvre.

For a step input

$$\delta(t') = \begin{cases} 0 & \text{for} \quad t' < 0 \\ \\ \delta_o & \text{for} \quad t' \geq 0 \end{cases}$$

and with zero initial states, the output vector of the system is given by

$$\begin{bmatrix} \beta(t') \\ r'(t') \end{bmatrix} = \int_{\tau=0}^{\tau=t'} [L][\phi(t'-\tau)][B]\,\delta(\tau)\,d\tau$$

$$= \delta_0 \int_{\tau=t'}^{\tau=0} \begin{bmatrix} \{A_{11}e^{-\chi_1(t'-\tau)} + C_{11}e^{-\chi_2(t'-\tau)}\} & \{A_{12}e^{-\chi_1(t'-\tau)} + C_{12}e^{-\chi_2(t'-\tau)}\} \\ \{A_{21}e^{-\chi_1(t'-\tau)} + C_{21}e^{-\chi_2(t'-\tau)}\} & \{A_{22}e^{-\chi_1(t'-\tau)} + C_{22}e^{-\chi_2(t'-\tau)}\} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} d(t'-\tau)$$

$$= -\delta_0 \begin{bmatrix} \left[\dfrac{B_1A_{11}+B_2A_{12}}{\chi_1}\right]\cdot e^{-\chi_1(t'-\tau)} + \left[\dfrac{B_1C_{11}+B_2C_{12}}{\chi_2}\right]\cdot e^{-\chi_2(t'-\tau)} \\ \left[\dfrac{B_1A_{21}+B_2A_{22}}{\chi_1}\right]\cdot e^{-\chi_1(t'-\tau)} + \left[\dfrac{B_2C_{21}+B_2C_{22}}{\chi_2}\right]\cdot e^{-\chi_2(t'-\tau)} \end{bmatrix}_{\tau=t'}^{\tau=0}$$

Finally,

$$\beta(t') = \delta_0 \left[\left[\frac{B_1A_{11}+B_2A_{12}}{\chi_1}\right]\cdot(1-e^{-\chi_1 t'}) + \left[\frac{B_1C_{11}+B_2C_{12}}{\chi_2}\right]\cdot(1-e^{-\chi_2 t'})\right]$$

$$r'(t') = \delta_o \left[\left[\frac{B_1A_{21}+B_2A_{22}}{\chi_1}\right](1-e^{-\chi_1 t'}) + \left[\frac{B_1C_{21}+B_2C_{22}}{\chi_2}\right](1-e^{-\chi_2 t'}) \right]$$

6. Conclusion

The problem of ship steering dynamics was analysed in the time domain using the modern technique of state space method, which is a powerful tool in the analysis, design and synthesis of complex, multi-variable, linear or nonlinear control systems.

The state and output matrix equations for the drift angle and yaw rate were derived, and displayed graphically by means of state variable diagram and state variable signal flow graph. Also the transfer matrix was expressed by this technique. A complete solution, by the state space analysis, for the drift angle and yaw rate was derived in the case of a turn manoeuvre with step rudder deflection. This procedure makes easy both digital and analog simulation of the problem.

7. References

[1] M.S. Denis and J.P. Craven, "Recent Contributions under the Bureau of Ship Fundamental Hydrodynamics Research Program (3-Control)", Journal of Ship Research, vol. 2, no.3, pp 1-22, December 1958.

[2] C.G. Källström and K.J. Åström, "Experiences of System Identification Applied to ship Steering", Automatica, vol. 17, no. 1, pp 187-198, 1981.

[3] J.P. Comstock (Ed.), Principles of Naval Architecture, SNAME, New York, 1974.

[4] J.P. Hooft, Advanced Dynamics of Marine Structures, John Wiley, New York, 1982.

[5] A. Ogawa and H. Kasai, "On the Mathematical Model of Manoeuvring

Motion of Ships", Int. Shipbuilding Progress, vol. 25, no. 292, pp 306-319, 1978.

[6] H. Eda and C.L. Crane, "Steering Characteristics of Ships in Calm Water and Waves", SNAME, vol. 73, pp 135-177, 1965.

[7] F.H. Raven, Automatic Control Engineering, 3rd Edition, McGraw-Hill, 1978.

[8] F. Csáki, State Space Methods for Control Systems, Akadémiai Kiadó, Budapest, 1977.

[9] K. Ogata, State Space Analysis of Control Systems, Prentice Hall Inc., New Jersey, 1967.