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A MODIFIED TODD-COXETER ALGORITHM

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Abstract

In respect to questions related to calculating the order of a group given by some presentation, presenting a subgroup of finite index in a finitely presented group and expressing any word of the group as a product of two elements, one of a complete set of coset representatives and the other a subgroup generator, we develop a technique for this purpose.

Notations

$G = < X,R>$ The group generated by the set of generators X and subjected to the set of relations R.

$[x,y]$ Equals $x^{-1} y^{-1} xy$ which is the commutator of the pair (x,y).

1. Introduction

In 1936 Todd and Coxeter described an algorithm for enumerating the cosets of a finitely generated subgroup of finite index in a finitely presented group. And since that date this algorithm represents one of the most important algorithms for the application in the field of group theory. Several authors ([1], [2], [3]) have discussed a modification of the algorithm to give a presentation of the subgroup in terms of the given generators.

In this paper a modified algorithm is applied to solve the following problem: let a group G be given by generators $g_1, g_2 ..., g_k$ and relations

$$R_1 (g_1,..., g_k) = ... = R_s (g_1, ... , g_k) = 1$$

and let a subgroup H of finite index in G be generated by a finite set of words $h_1, ..., h_p$ in the generators of G and its inverses, then

1) Find a set of representatives $y_1,..., y_t$ of the right cosets of H in G.
2) Express any word W satisfying 1W=1, so it is an element of the subgroup, as a word in the generators of H.

3) More generally, express any element of the group G as a product WB, where W is a word in the subgroup generators and B is a coset representative.

2. The algorithm

It is assumed that the reader is familiar with the Todd-Coxeter algorithm as given in [5] and [6]. Consider the problem as formulated in the introduction. If the subgroup H is of finite index n in G, then the Todd-Coxeter algorithm will yield a complete augmented coset table, i.e. a table with n rows and 2k columns corresponding to the generators of G and its inverses. Moreover, a set of coset representatives $y_1, \ldots, y_t$ can be obtained as will be discussed after the next definition.

Definition

For any two coset representatives K_i and K_j, we define the mapping φ by setting $j\varphi = (i,r)$, where this is the first pair such that $K_i g_r = K_j$.

The cosets $K_1, \ldots, K_n$ can be defined inductively from the φ mapping of the coset table, by setting $K_1 = H$ and $K_j = K_i g_r$, where $j\varphi = (i,r)$.

3. The Word Problem

For any cost representative 1 and some word $W = x_1 x_2 \ldots x_q$ in the generators of G, the extract of the sequence of the coset numbers $1_1 = 1$, $1_2 = 1_1 x_1, \ldots, 1_{r+1} = 1_r x_r$ is called the trace of the word

W from the cost 1 through the augmented coset table.

The basic operation for dealing with the word problem is the technique of the word tracing. In our word problem, on tracing the word $W(g_i)$ from the coset representative 1 and then substitute from the coset table, we can write

$$1\, W(g_i) = V_{1,w}\,(h_j)\, K$$

where K is a coset representative and $V_{1,w}$ is a word in the generators of the subgroup h_j, resulting from tracing the word $W(g_i)$ from coset 1 through the augmented coset table.

4. A worked Example

Let the group G be presented by

$$G = < X, Y \;\; X^8 , Y^2, X^y = X^2 >$$

and let the subgroup H be generated by X^2.
The one-row subgroup table will be

	A^{-1}		X		X	
1		1		2		1

Define 2 = 1X, we get the deduced entry
$2X = A1$, $1X^{-1} = A^{-1}\, 2$ from the subgroup table.
Continue, define 3 = 1Y we get the deduced entry

$$3Y = E1,\; 1Y^{-1} = E3 \text{ from row 1 of relation II}$$

Defining 4 = 2Y = XY yields the information

$4Y = E2$, $2Y^{-1} = E4$ from row 2 of relation II

On inserting these entries in the rows of relation III, we get the following deduced entries:

$3X = 3Y\,X\,X\,X\,Y^{-1} = A\;4$, $4X^{-1} = A^{-1}\;3$ from row 1,

$4X = 4Y\,X\,X\,X\,Y^{-1} = A^{2}\;3$, $3X^{-1} = A^{-2}\;4$ from row 2,

At this stage, all the tables have its final form as:

*	X	Y	X^{-1}	Y^{-1}	
1	E 2	E 3	A^{-1} 2	E 3	
2	A 1	E 4	E 1	E 4	$2\varphi = (1, 1)$
3	A 4	E 1	A^{-2} 4	E 1	$3\varphi = (1, 2)$
4	A^{2} 3	E 2	A^{-1} 3	E 2	$4\varphi = (2, 2)$

"The augmented coset table"

	X	X	X	X	X	X	X	X
1	2	1	2	1	2	1	2	1
2	1	2	1	2	1	2	1	2

I

	Y	Y
1	3	1
2	4	2

II

	Y^{-1}	X	Y	X^{-1}	X^{-1}	X^{-1}
1	3	4	2	1	2	1
2	4	3	1	2	1	2

III

" The relation tables"

The coset representatives are:

$y_1 = e,\ y_2 = X\ ,\ y_3 = Y\ ,\ y_4 = XY$

The word $W_1 = [X,Y]$ can be expressed in the form $V_{1,w_1}K$ follows:

$$
\begin{aligned}
1\ W_1 &= 1X^{-1}\ Y^{-1}\ X\ Y \\
&= A^{-1}\ 2Y^{-1}\ X\ Y \\
&= A^{-1}\ 4\ X\ Y \\
&= A\ 3\ Y \\
&= A\ 1
\end{aligned}
$$

Since the tracing of W_1 gives A1, this shows that the word W_1 is an element of the subgroup so we can express it in terms of the subgroup generators only as:

$$W_1 = A = X^2 \varepsilon\ H$$

Similarly, we can express the word $W_2 = X^y\ [X\ ,\ Y]$ in the form

$$
\begin{aligned}
1W_2 &= 1\ Y^{-1}\ X\ Y\ X^{-1}\ Y^{-1}\ X\ Y \\
&= E\ 3X\ Y\ X^{-1}\ Y^{-1}\ X\ Y \\
&= A\ 4\ Y\ X^{-1}\ Y^{-1}\ X\ Y \\
&= A\ 2X^{-1}\ Y^{-1}\ X\ Y \\
&= A\ 1Y^{-1}\ X\ Y \\
&= A\ 3X\ Y \\
&= A^2\ 4\ Y \\
&= A^2\ 2
\end{aligned}
$$

$$\therefore\quad 1W_2 = X^4\ 2 \longrightarrow W_2 = X^4 * X = X^5$$

which gives us the information that the word W_2 is found in coset 2.

Now, we can express all the elements of the group G in terms of the subgroup generators and the coset representatives . This is done by forming the elements of each coset as follows:

$$\text{Coset No. } 1 = \{ e, X^2, X^4, X^6\}\ y_1 = \{e, X^2, X^4, X^6 \}$$

$$\text{Coset No. } 2 = \{ e, X^2, X^4, X^6\}\ y_2 = \{X, X^3, X^5, X^7 \}$$

$$\text{Coset No. } 3 = \{ e, X^2, X^4, X^6\}\ y_3 = \{Y, X^2Y, X^4Y, X^6 Y \}$$

$$\text{Coset No. } 4 = \{ e, X^2, X^4, X^6\}\ y_4 = \{XY, X^3Y, X^5Y, X^7Y \}$$

and thus the sixteen elements of the given group are all the elements contained in the above four cosets. Each of them is expressed as the product of a coset representative and one of the subgroup elements.

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