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# THE MONENT GENERATING FUNCTION OF PILLAI'S CRITERION CONCERNING THREE HYPOTHESES UNDER VIOLATIONS IN THE COMPLEX CASE

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**Abstract**

The moment generating function of Pillai's criterion $V^{(p)}=\mathrm{tr}\, R(I+R)^{-1}$ $R=S_1^{\frac{1}{2}} S_2^{-1} S_1^{\frac{1}{2}}$ is derived under violation in the complex case in connection with (a) test of $\Sigma_1=\Sigma_2$ of two complex normal populations $N_c(\mu_i, \Sigma_i)$, i=1,2 and (b) MANOVA. The derivation is based on the complex joint pdf of $r_1, r_2, \ldots, r_p$ which is obtained by Pillai and Hsu (1979) under violation, where $S_1$ and $S_2$ are inependently distributed according to complex noncentral Wishart $W_c(P.n_1,\Sigma_1,\Omega)$ and central $W_c(p,n_2,\Sigma_2, 0)$. We considered also the special cases ($\Omega=0$) and($\Sigma_1=\Sigma_2$), the forms obtained are the complex analogue of those obtained by Khatri (1967) and Pillai (1968) in the real cases. The results were further extended to include the test (c) canoniocal correlation, this is carried out under the assumption that $\Omega$ is random. The findings may facilitate the derivation of lower nocentral moments of the criteria.

## I. Introduction

The moment generating function of Pillai's criterion $V^{(p)}$ has been studied by several authors. Khatri (1967) has obtained the mgf of $V^{(p)}$ in the case of testing the equality of two covariance matrices of two normal populations. While the mgf of the criterion has been also considered by Pillai (1968) in connection with NANOVA and canonical correlation in the real cases. The mgf's obtained by Khatri-Pillai are an extension of the earlier results obtained by Pillai (1954), (1956) and by James (1964) in the centeral case.

Now we shall consider the following hypotheses:

(a) Test of equality of two covariance matrices of two complex normal Populations,

(b) Test of equality of mean vectors of 1 p-complex normal populations (MANOVA).

In connection with (a) and (b), Pillai and Hsu (1979) have obtained the joint pdf of the latent roots $r_1, r_2, \ldots, r_p$ of $R = S_1^{\frac{1}{2}} S_2^{-1} S_1^{\frac{1}{2}}$ under violation in the complex case, it is given here by eqn. (3.1). Based on this density function, we derived two forms for the mgf of Pillai's criterion $V^{(p)} = \mathrm{tr}\, S_1(S_1 + S_2)^{-1} = \mathrm{tr}\, R(I+R)^{-1}$, and $R = \mathrm{diag}\,(r_1, r_2, \ldots, r_p), 0 \leq r_1 \leq r_2 \ldots, \leq r_p \leq 1$, in connection with the hypotheses (a) and (b) under violation, where the expressions are given in Theorem 3.1. In section 4, we considered the m.g.f. of $V^{(p)}$ when (i) $\Omega = 0$ and (ii) $\Sigma_1 = \Sigma_2$.

In order to derive the mgf of $V^{(p)}$ in connection with (c) test of independence of a p set and q set in a (p+q) - complex multivariate normal (canonical correlation), the results of section (3) are extended in Section 5. By using Theorem 3.1 and considering the matrix $\Omega$ completely random (see Pillai (1975) and Constantine (1963)), then we obtained two forms of $V^{(p)}$ in the canonical correlation case under violation (Theorem 5.1). It was verified that the complex analogue of eqn (4.3) of Pillai (1968) may follow from Theorem 5.1 as a special case.

## 2. Preliminaries

In this section we shall outline some results concerning functions of a Hermitian matrix and complex zonal polynomials.

Definition 2.1. The complex zonal polynomial of degree K of a hermitian matrix $\tilde{C}_k(W)$ is as defined by James (1964), expressible in terms of certain homogeneous symmetric polynomial in the characteristic roots of W. It is noted that $k = (k_1, k_2, \ldots, k_p)$ is a partition of $k$ into p $k_1 \geq k_2, \ldots k_p \geq 0$ and $\sum_{i=1} k_i = k$.

Lemma 2.1. Let W be positive definite Hermitian matrix, hence due to Khatri (1969) we have

$$\int_{I=W=\bar{W}'>0} |W|^{a-p}\, |I-W|^{b-p}\, \tilde{C}_k(W)dW = \frac{\tilde{\Gamma}_p(a,k)\, \tilde{\Gamma}_p(a)\, \tilde{C}_k(I)}{\tilde{\Gamma}_p(a+b,k)} \qquad (2.2)$$

where $\tilde{\Gamma}_p(a,k)$, $\tilde{\Gamma}_p(a)$ are the multivariate gamma functions. Since W being positive definite function, it may be transformed by unitary matrix $U(U\bar{U}'=I)$, so that UWU' is diagonal. Taking into consideration the jacobian of transformation which is given by Khatri (1965) in the form

$$J = \prod_{i>j} (w_i - w_j)^2 h_2(U), \quad \text{and} \quad \int_{U\bar{U}'=I} h_2(U) = \frac{\pi^{\frac{1}{2}p(p-1)}}{\tilde{\Gamma}_p}$$

Hence, we get from (2.2) the following result:

$$\int_{I=\bar{W}'=W>0} |W|^{a-p} \, |I-W|^{b-p} \, \tilde{C}_k(W) \prod_{i>j} (w_i - w_j)^2 \, dW$$

$$= \frac{\Gamma_p(p) \; \Gamma_p(a,k) \; \Gamma_p(b) \; \tilde{C}_k(I)}{\pi^{p(p-1)} \; \tilde{\Gamma}_p(a+b,k)} \tag{2.3}$$

Lemma 2.2 Let A and B be two Hermitian matrices, hence we have

$$\int_{A=\bar{A}'>0} e^{-trA} \, |A|^{a-p} \tilde{C}_k(AB) \, dA = \tilde{\Gamma}_p(a) \, [a]_k \tilde{C}_k(B) \tag{2.4}$$

This is eqn (86) of James (1964).

Lemma 2.3. If Z and R positive definite Hemitian matrices, hence we have the following integral

$$\frac{2^{p(p-1)}}{(2\pi i)^{p^2}} \int_{ReZ>0} e^{trZ} \, |Z|^{-n} \, C_k(Z^{-1}R) dZ = \frac{\tilde{C}_k(R)}{\tilde{\Gamma}_p(n,k)} \tag{2.5}$$

This is an immediate result of the inverse Laplace transform (Herz(1955)

Definition 2.3 Complex Laguere P olynomials

In the complex case, the Laguerre polynomials are defined by Khatri (1970) in the form

$$L_k^{\gamma}(S) = e^{trS} \int_{R>o} [\tilde{\Gamma}_p(\gamma+p)]^{-1} \, {}_0\tilde{F}_1(\gamma+P, -RS) e^{-trR} |R|^{\gamma} \tilde{C}_k(R) dR \tag{2.6}$$

Where S is an arbitrary pxp Hermitian matrix and $_0\tilde{F}_1$ is the complex hypergeometric function as in equation (87) of James (1964). In view of Constantine (1966) an equivalent expression for $\tilde{L}_k^{\gamma}(S)$ will be

$$\tilde{L}_k^{\gamma}(S) = \frac{2^{p(p-1)}}{(2\pi i)^p} \tilde{\Gamma}_p(\gamma+p, k) \int_{ReZ>0} e^{trZ} |Z|^{-\gamma-p} \tilde{C}_k(I-Z^{-1}S)dZ \qquad (2.7)$$

To express $\tilde{L}_k^{\nu}(S)$ in a more convenient form, the following relationship is needed

$$\frac{\tilde{C}_k(I-S)}{\tilde{C}_k(I)} = \sum_{n=0}^{K} \sum_{\nu} \frac{(-1)^n \tilde{a}_{k\,\nu} \tilde{C}_{\nu}(S)}{\tilde{C}_{\nu}(I)} \qquad (2.8)$$

$\tilde{a}_{k,\nu}$ are constants and $\nu$ is a partition of n. Upon using (2.8), integrating term by term by the aid of (2.5), hence from (2.7) we get:

$$\tilde{L}_k^{\gamma}(S) = (\gamma+p)_k \tilde{C}_k(I) \sum_{n=0}^{K} \sum_{\nu} \frac{(-1)^n \tilde{a}_{k',\nu} \tilde{C}_{\nu}(S)}{(\gamma+p)_{\nu} \tilde{C}_{\nu}(I)} \qquad (2.9)$$

## 3. The Moment Generating Function of Pillai's Criterion in Connection with Two Multivariate Hypotheses

Let $R = \text{diag}(r_1, r_2, \ldots, r_p)$ where $0 \leq r_1 \leq \ldots r_p <$ are the roots of $R = S_2^{-\frac{1}{2}} S_1 S_2^{-\frac{1}{2}}$, where $S_1$ and $S_2$ are distributed as before. The joint pdf of $r_1, \ldots, r_p$ has been obtained by Pillai and Hsu (1979) under violation, it is in the form:

$$C(p,n_1 n_2)e^{-tr\Omega} |\Lambda|^{-n_1} |R|^{n_1-p} |I+\lambda R|^{-(n_1+n_2)} \prod_{i>j} (r_i-r_j)^2$$

$$\sum_{K=0}^{\infty} \sum_{k} \frac{[n_1+n_2]_k \tilde{C}_k[\lambda R(I+ R)^{-1}]}{K!}$$

$$\sum_{n=0}^{K} \sum_{\nu} \frac{(-\lambda^{-n})\tilde{a}_{k,\nu} \tilde{C}_{\nu} (\Lambda^{-1})\tilde{L}_{\nu}^{(n_1-p)}(\Omega)}{\tilde{C}_{\nu}(I)\tilde{C}_{\nu}(I)[n_1]_{\nu}} \qquad (3.1)$$

where

$$C(p,n_1,n_2)= \pi^{p(p-1)}\Gamma_p(n_1+n_2)/[\Gamma_p(n_1)\Gamma_p(n_2)\Gamma_p(p)] \qquad (3.2)$$

$\tilde{a}_{k,\nu}$ are constants defined by (2.8), $\nu$ is a partition of n and $\tilde{L}_{\nu}^{(n_1-p)}(\Omega)$ is the generalized Laguerre polynomial. The derivation of (3.1) is carried out under the assumption that $\Lambda = \Sigma_1 \Sigma_2^{-1}$ is "random".

Next, we shall obtain two expressions for the m.g.f. of $V^{(p)} = trR(I+R)^{-1}$ concerning the hypotheses: (a) Test of $\Sigma_1 = \Sigma_2$ of two complex normal populations, (b) MANOVA.

The derivation of the m.g.f. of $V^{(p)}$ is based on the joint p.d.f. as given by eqn.(3.1). Consider the transformation $l_i = r_i(1+r_i)^{-1}$ or $L=R(I+R)^{-1}$, hence from (3.1) we obtain the joint density function of $l_1, l_2, \ldots, l_p$ in the form:

$$C(p,n_1,n_2)\, e^{-tr\,\Omega}|\Lambda|^{-n_1}|L|^{n_1-p}|I-L|^{n_2-p}$$

$$\prod_{i>j}^{p} (l_i - l_j)^2 \sum_{K=0} \sum_{k} \frac{[n_1+n_2]_k \tilde{C}_k(L)}{K!} \times E(\Lambda^{-1}, \Omega) \qquad (3.4)$$

$C(p,n_1,n_2)$ as in (3.2) and,

$$E(\Lambda^{-1}, \Omega) = \sum_{n=o}^{K} \sum_{\nu} \frac{(-\lambda^{-n})\tilde{a}_{k,\nu} \tilde{C}_\nu(\Lambda^{-1})}{\tilde{C}_\nu(I)\, \tilde{C}_\nu(I)\, [n_1]_\nu} \tilde{L}_\nu^{(n_1-p)}(\Omega) \qquad (3.5)$$

Starting from here, we may proceed by two different techniques to obtain two forms for the m.g.f. of $V^{(p)}$

**Form (i)** In view of (2.7), we may write (3.5) as:

$$E(\Lambda^{-1}, \Omega) = \sum_{n=0}^{K} \sum_{\nu} \frac{\tilde{a}_{k,\nu} \tilde{C}_\nu(-\lambda^{-1}\Lambda - 1)}{\tilde{C}_\nu(I)\; \tilde{C}_\nu(I)} \times \tilde{\Gamma}_p(n_1) \qquad (3.6)$$

$$\frac{2^{p(p-1)}}{(2\pi i)^{p^2}} \int_{ReZ>0} e^{tr\, Z} |Z|^{-n_1} \tilde{C}_\nu(I - Z^{-1}\Omega)\, dz$$

We may note that the splitting formula of zonal polynomials is given by equation (92) of James (1964). It is as follows,

$$\int_{O(P)} \tilde{C}_\nu(AHTH')(dH) = \frac{\tilde{C}_\nu(A)\, \tilde{C}_\nu(B)}{\tilde{C}_\nu(I_p)} \qquad (3.7)$$

where (dH) stands for the invariant Haar measure on the orthogonal group O(p). Upon using (2.8), applying the splitting formula, and by virtue of

Lemma (2.2), it is possible to rewrite (3.4) as:

$$\frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \cdot \frac{2^{p(p-1)}}{(2\pi i)^{p^2}} \frac{1}{\tilde{\Gamma}_p(n_2)} e^{-tr\,\Omega} |\lambda\Lambda|^{-n_1} |L|^{n_1-p}$$

$$|I-L|^{n_2-p} \prod_{i>j}(l_i-l_j)^2 \int_{ReZ>0} e^{tr\,Z} |Z|^{-n_1} \int_{S>0} e^{-tr\,S} |S|^{n_1+n_2-p}$$

$$\exp[S\{I-\lambda^{-1}\Lambda^{-\frac{1}{2}}(I-Z^{-1}\Omega)\Lambda^{-\frac{1}{2}}\}L]\, dSdZ$$

Then multiply by $e^{t\,trL}$, and perform the integration w.r.t. $L=\tilde{L}'$, from 0 to I by using (2.3). Hence, by successive application of (2.8), integrating w.r.t. S using (2.4) and by applying (2.8) we have:

$$M(t) = \frac{2^{p(p-1)}}{(2\pi 1)p^2} x\, \tilde{\Gamma}_p(n_1) e^{-tr\Omega} |\lambda\Lambda|^{-n_1} \sum_{K=0} \sum_k \frac{[n_1]k}{[n_!+n2]_k}$$

$$x\, \frac{\tilde{C}_k(I)}{K!} \sum_{d=o}^{k} \sum_{\delta} t.^{K-d}\, \tilde{a}_{k,\delta}\, [n_1+n_2]_\delta \sum_{n=o}^{d} \sum_{\nu} \frac{\tilde{a}_{\delta\,\nu}}{\tilde{C}_\nu(I)}\, x$$

$$x \int_{ReZ>o} e^{tr\Omega} |Z|^{-n_1} \tilde{C}_\nu(-\lambda^{-1}\Lambda^{-\frac{1}{2}}(I-Z^{-1}\Omega)\Lambda^{-\frac{1}{2}})\, dZ \qquad (3.8)$$

Noting that the integral in (3.8) is in fact the integral representation (2.7) of Laguerre polynomial, therefore, the final form of M(t) will be as in form (i) of Theorem 3.1.

The alternative form of the m.g.f. of $V^{(P)}$ may be obtained as follows:

## 4. Special Cases

i- when $\Omega = 0$. Substituting $\Omega = 0$ into (3.10), noting that $\tilde{L}_k^{\gamma}(0) = (\gamma + p)_k C_k(I)$ and by using (2.8) we get:

$$M(t) = |\lambda\Lambda|^{-n_1} \sum_{K=o} \sum_k \frac{[n_1]_k}{[n_1+n_2]_k} \cdot \frac{\tilde{C}_k(I)}{K!}$$

$$\sum_{d=o}^{K} \sum_{\delta} \frac{t^{k-d}\, \tilde{a}_{k,\delta}\, [n_1+n_2]_\delta\, \tilde{C}_\delta\, (I - \lambda^{-1}\Lambda^{-1})}{\tilde{C}_\delta\,(I)} \tag{4.1}$$

This result is the complex analogues of eqn. (14) of Khatri (1967) Similarly, upon specializing $\Omega = 0$ in (3.11), then

$$M(t) = |\lambda\Lambda|^{-n_1} \sum_{K=o} \sum_k \frac{[n_1+n_2]_k}{K!} \cdot \frac{\tilde{C}_k(I - \lambda^{-1}\Lambda^{-1})}{\tilde{C}_k(I)}$$

$$x \sum_{i=o} \frac{t^i}{i!} \left( \sum_{\lambda,\eta} g^{n}_{k,\lambda} \frac{[n_1]_n}{[n_1+n_2]_\eta} \tilde{C}_\eta(I) \right) \tag{4.2}$$

ii. when $\lambda = 1$, $\Lambda = I$. Upon replacing $(\lambda = 1, \Lambda = I)$ in (3.10), we get

$$e^{-tr\,\Omega} \sum_{K=o} \sum_k \frac{[n_1]_k}{[n_1+n_2]_k} \cdot \frac{\tilde{C}_k(I)}{K!} \, x \sum_{d=o}^{K} \sum_{\delta} t^{k-d} \tilde{a}_{k,\delta} \frac{[n_1+n_2]_\delta}{[n_1]} \, x \frac{\tilde{C}_\delta(\Omega)}{\tilde{C}_\delta(I)} \tag{4.3}$$

So, eqn. (4.3) is analogous to Pillai's expression of $V^{(p)}$ in the real case, also from (3.11), we get

$$M(t) = e^{-tr\,\Omega} \sum_{K=o} \sum_k \frac{[n_1+n_2]_k}{[n_1]_k} \cdot \frac{\tilde{C}_k(\Omega)}{\tilde{C}_k(I) K!} \, x \sum_{i=o} \frac{t^i}{i!} \sum_{\lambda\,\eta} \tilde{g}^{\eta}_{k,\lambda} \frac{[n_1]_\eta}{[n_1+n_2]_\eta} \tilde{C}_\eta(I) \tag{4.4}$$

## 5. The Moment Generating Function of $V^{(p)}$ in the Canonical Correlation case under violation in the complex case.

To derive the mgf of $V^{(p)}$, we shall assume $\Omega$ random matrix (see constantine (1963) and Pillali (1975), i.e. $\Omega = \sum_1^{\frac{1}{2}} MYY'M' \sum_1^{-\frac{1}{2}}$ , where YY' has a complex central Wishart $W_c(p,n_3,\sum_3,0)$, $n_3=n_1+n_2$ . The pdf of YY' is given by,

$$\{\tilde{\Gamma}_q(n_3)|\Sigma_3|\}^{-1} e^{-tr\,\Sigma_3^{-1} YY'} |YY'|^{n_3-q} \qquad (5.1)$$

Start from (3.10), use the series expansion of $\tilde{L}_\nu^{n_1-p}(\Omega)$ and multiply by (5.1). After integrating w.r.t. YY' using (2.4), we get after some simplification, form (i). The second form of $V^{(p)}$ may be obtained like manner from (3.11).

**Theorem 5.1:** The m.g.f. of $V^{(p)}$ in the canonical correlation case under violation in the complex case is given by:

$$\text{(i)}\quad M(t) = |\lambda\Lambda|^{-n_1}|I-P^2|^{(n_1+n_2)} \sum_{K=o}\sum_{k} \frac{[n_1]_k}{[n_1+n_2]_k} \times \frac{\tilde{C}_k(I)}{K!}$$

$$\times \sum_{d=o}^{K}\sum_{\delta} t^{k-d}\, \tilde{a}_{k,\delta}\,[n_1+n_2]_\delta \sum_{n=o}^{d}\sum_{\nu} \frac{\tilde{a}_{\delta,\nu}\,\tilde{C}_\nu(-\lambda^{-1}\Lambda^{-1})}{\tilde{C}_\nu(I)}$$

$$\times \sum_{i=o}^{n}\sum_{\eta} \frac{(-1)^i\,\tilde{a}_{\nu,\eta}[n_1+n_2]_\eta}{[n_1]_\eta\,\tilde{C}_\eta(I)} \tilde{C}_\eta(P^2), \qquad (5.2)$$

$$P^2 = (I+\Omega_1)^{-1}\Omega_1;\ \Omega_1 = \sum_3^{+\frac{1}{2}} M' \sum_1^{-1} M \sum_3^{\frac{1}{2}}$$

$$\text{(ii)}\quad M(t) = |\lambda\Lambda|^{-n_1}|I-P^2|^{(n_1+n_2)}$$

$$\times \sum_{K=o}\sum_{k} \frac{[n_1+n_2]_k}{K!} \sum_{i=o} \frac{t^i}{i!} \Big(\sum_{\lambda,\eta} \tilde{g}^{\eta}_{k,\lambda} \frac{[n_1]\,\tilde{C}_\eta(I)}{[n_1+n_2]_\eta}\Big)$$

$$\sum_{n=o}^{k} \sum_{\nu} \frac{\tilde{a}_{k,\nu} \tilde{C}_{\nu} (-\lambda^{-1} \Lambda^{-1})}{\tilde{C}_{\nu} (I)}, \sum_{d=o}^{n} \sum_{\delta} \frac{(-1)^{d} \tilde{a}_{\nu,\delta} [n_1+n_2]_{\delta}}{[n_1]_{\delta} \tilde{C}_{\delta}(I)} \times \tilde{C}_{\delta} (P^2) \quad (5.3)$$

**Special Case:** Upon replacing $\lambda = 1$, $\lambda = I$ in equation (5.2), then

$$M(t) = |I-p^2|^{(n_1+n_2)} \sum_{K=o} \sum_{k} \frac{[n_1]_k}{[n_1+n_2]_k} \cdot \frac{\tilde{C}_k(I)}{K!}$$

$$\times \sum_{d=o}^{K} \sum_{\delta} t^{k-d} \tilde{a}_{k,\delta} \frac{([n_1+n_2]_{\delta})^2}{[n_1]_{\delta} \; C_{\delta} (I)} \tilde{C}_{\delta} (P^2)$$

This result is an extension to that obtained by Pillai (1968).

Again from equation (5.3) we have:

$$| I-P^2 |^{n_1+n_2} \sum_{K=o} \sum_{k} \frac{([n_1+n_2]_k)^2}{[n_1]_k} \times \frac{\tilde{C}_k(P^2)}{\tilde{C}_k(I) \; K!}$$

$$\times \sum_{i=o} \frac{t^i}{i!} (\sum_{\lambda,\eta} \tilde{g}^{\lambda}_{k,\lambda} \frac{[n_1]_{\eta} \tilde{C}_{\eta} (I)}{[n_1+n_2]_{\eta}}$$

## Notations

| | | |
|---|---|---|
| $\lvert A \rvert$ | = | determinant of a matrix A |
| tr A | = | trace of a matrix A |
| Re A | = | real part of A |
| $\Gamma_p(n)$ | = | complex multivariate gamma function |
| MANOVA | = | multivariate analysis of variance. |

**References**

[1] Constantine, A.G. (1963). Some noncentral distribution problems in multivariate analysis. Ann. Math. Statist. 34 1270-1285.

[2] Constantine, A.G. (1966). The distribution of Hotelling's generalized $T_0^2$. Ann. Math. Statist. 37 217-225.

[3] Herz C.S. (1955). Bessel functions of matrix argument. Ann. of Math. 61 474-523.

[4] James, A.T. (1964). Distribution of matrix variates and latent roots derived from normal samples. Ann. Math. Statist. 35 475-501.

[5] Khatri, C.G. (1965). Classical Statistical Analysis based on certain multivariate complex Gaussian distribution. Ann. Math. Statist. 36 98-114.

[6] Khatri, C.G. (1967). Some distribution problems connected with the characteristic roots of $S_1S_2^{-1}$. Ann. Math. Statist. 38 944-948

[7] Khatri, C.G. (1969). Non-central distributions of the ith largest characteristic roots of three matrices concerning complex multivariate normal populations. Ann. Inst. Statist. Math. 21 23-32

[8] Khatri, C.G. (1970). On the moments of traces of two matrices in three situations for complex multivariate normal populations. Sankhya 32 65-80.

[9] Khatri C.G. and Pillai, K.C.S. (1965). Some results on the noncentral multivariate beta distribution and moments of traces of two matrices. Ann. Math. Statist. 36 1511-1520.

[10] Khatri. C.G. and Pillai. K.C.S. (1967). On the moments of races of two matrices in multivariate analysis. Ann. Inst. Statist. Math. 19 143-156.

[11] Khatri, C.G. and Pillai, K.C.S. (1968). On the noncentral distributions of two test criteria in multivariate analysis of variance. Ann. Math. Statist. 39 215-226.

[12] Pillai, K.C.S. (1954). On some distribution problems in multivariate analysis Mimeo. Series No. 88, Institute of Statistics, Univ. North Carolina Chapel Hill.

[13] Pillai. K.C.S. (1956). Some results useful in multivariate analysis. Ann. Math. Statist. 27 1106-1114.

[14] Pillai, K.C.S. (1968). On the moment generating function of Pillai's $V^{(s)}$ criterion. Ann. Math. Statist. 39 877-880.

[15] Pillai. K.C.S. (1975). The distribution of the characteristic roots of $S_1 S_2{-1}$ under violations. Ann. Statist. 3 773-779.

[16] Pillai, K.C.S. and Hsu, Y. (1979). The distribution of the characteristic roots of of $S_1S_2^{-1}$ under violations in the complex case and power comparisons of four tests. Ann. Inst. Statist. Math. 31 445-463.