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PERSISTING RADIATION OF Φ^4-SOLITARY SOLUTION WHEN IT PASSES A PERTURBATION

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ABSTRACT

One dimensional solitary solution of the nonlinear Φ^4-wave equation behaves like Newtonian particle if subjected to external forces. We observe here that they also emit free radiation (persisting radiation) if accelerated by an external force. We also calculate the radiation spectrum and the energy distribution among the different excited modes.

NOTATIONS

H	=	Total Hamiltonian density
H_p	=	Hamiltonian density of impurity
H_o	=	Hamiltonian density without impurity
V	=	Impurity potential
χ	=	Amplitude of impurity potential
β		Dimensionless velocity of the kink
x	=	Dimensionless coordinate
t	=	Dimensionless time
ω_T	=	Eigen-frequency of translation mode
ω_L	=	,, ,, ,, local mode
ω_q	=	,, ,, ,, scattering mode
ω	=	Fourier variable
δ	=	Dirac delta function
θ	=	Heaviside step function
F	=	Force.
V_g	=	Group velocity
E_r	=	Radiation energy
P_r	=	Radiation momentum.
E_L	=	Excitation energy of the local mode
ϕ^k	=	Kink solution
$f_i(x)$	=	Normal modes
φ^v	=	Stationary field dressing of the impurity
$\varphi_L(\tau)$	=	Amplilude of local responce
$\varphi_L^s(\tau)$	=	Steady state amplilude of local responce
$\varphi_q(\tau)$	=	Amplitude of scattering mode
$\varphi_q^r(\tau)$	=	Radiation amplitude

1. INTRODUCTION

Particle like solitons of nonlinear wave equations known as solitons, have attracted recently considerable attention of physicists, engineers and mathematicians [1]. Although restricted to two space time dimensions solition-bearing equations are considered to yield very interesting mathematical models to such diverse objects elementary particles, transmission lines, Bloch walls, phase transitions and various other excitations. The interaction of solitary waves with spatial inhomogeneities is of considerable importance for all the above applications.

We use a recently developed perturbation theory [2,3] to treat such interactions to lowest order in external forces. It is shown [2,4] that solitary solution behaves like classical particle which obey Newton's law.

The solitary wave equations also admit excitations of small amplitude in the field quantity which the solitary wave represents, for instance, spin waves (magnons) in Bloch wall system and lattice vibrations (phonons) in phase transitions. These solution, in the abscence of solitary solution and presence of perturbations, of the linearized field equation may be called vacuum response.

Upon interaction with a perturbation the solitary solution will dress itself with transient fields. One is led to expect that the solitary solution must also emit persisting radiation when it passes a perturbation.

Moreover in the case of the ϕ^4 field equation [5], the solitary solution (kink) undergoes deformations (internal oscilations).

We are mainly interested in the spectrum of persisting radiation, local and translational variations of the kink upon interacting with spatial perturbations.

In the next section we recall basic formulas and introduce our notations. In section (3) we treat the effect of an external force on the kink with emphasis on the radiative and local part and we derive the basic expression for the spectral distribution of the radiation. We apply this formula in section (4) in some limiting cases.

2. THE MODEL

In this section we list a few formulas and introduce our notation. We consider the classical field theory described by the Hamiltonian densities :

$$H = H_o + H_p \tag{2.1}$$

$$H_o = R \left[\frac{1}{2} \left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{4} \left(\frac{\partial\psi}{\partial x}\right)^2 - \frac{1}{2} \psi^2 + \frac{1}{4} \psi^4 \right] \tag{2.2}$$

$$H_p = - \psi F(x,t) = \frac{\partial\psi}{\partial x} V(x,t) \tag{2.3}$$

We shall assume for V(x,t) the impurity potential

$$V(x) = - \chi \, [\, \theta(x + x_o) - \theta(x - x_o) \,] \quad , \tag{2.4}$$

where $\theta(x)$ is the Heaviside step function. For the force we get

$$F(x) = - \chi \, [\, \delta(x+x_o) - \delta(x-x_o) \,] \quad , \tag{2.5}$$

where δ is the Dirac delta function. The kink solution are known to be [2,6].

$$\psi^k(x,t) = \tanh \gamma(x - \beta t) \quad , \tag{2.6}$$

where

$$\gamma = (1 - 2 \beta^2)^{-\frac{1}{2}} \tag{2.7}$$

and β is the dimensionless velocity of the kink. The normal modes f_i (x) of infiniteimal deviations from the stationary ($\beta = 0$) kink satisfy the equation

$$[- \frac{1}{2} \frac{d^2}{dx^2} - (2 - \frac{3}{\cosh^2 x})] f_i(x) = \omega_i^2 f_i(x) \quad , \tag{2.8}$$

and from a complete, orthonormal basis in which we may expand our perturbation. The set of function is given by

(i) the translation mode:

$$f_T(x) = (\frac{3}{4})^{\frac{1}{2}} \frac{1}{\cosh^2 x} \quad ; \quad \omega_T = 0 \quad , \tag{2.9}$$

(ii) the local mode:

$$f_L(x) = \left(\frac{3}{2}\right)^{\frac{1}{2}} \frac{\sinh x}{\cosh^2 x} \quad ; \omega_L^2 = 3/2 \quad , \tag{2.10}$$

and **(iii) the scattering or continuum mode:**

$$f_q(x) = \frac{e^{i q x}}{[2\pi(4+5q^2+q^4)]^{1/2}} [\ 3 \tanh^2 x - 1 - q^2 - 3iq \tanh x] \ ;$$

$$\omega_q^2 = q^2/2 + 2 \ . \tag{2.11}$$

3. LOCAL VARIATIONS AND DERIVATION OF THE BASIC RADIATION FORMULA

We shall now solve, to linear order in χ, the equation

$$\frac{\partial^2\psi}{\partial t^2} - \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - \psi + \psi^3 = - \chi\ [\ \delta(x+x_o) - \delta(x-x_o)\ ] \tag{3.1}$$

as an initial value problem. The initial condition as $t \rightarrow \infty$ corresponds to the physical situation consisting of a kink far from the impurity, moving towards it and the stationary field dressing of the impurity which is

$$\phi^v(x) = - \frac{\chi}{2} [\ e^{-2|x+x_o|} - e^{-2|x-x_o|}\ ] + o(\chi) \tag{3.2}$$

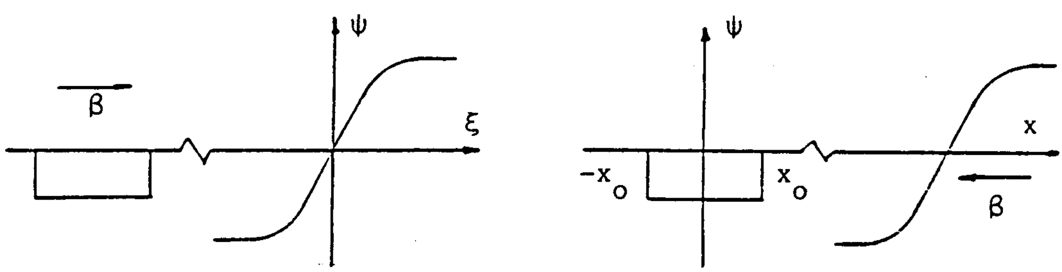
We expect to find as $t \rightarrow \infty$ the same state at the site of the impurity, and a kink moving towards infinity with changed velocity and persisting radiation. It is almost impossible, however, to calculate this directly in the restframe of the impurity; one does the calculation rather in the rest frame of the kink and change to the laboratory frame only at the end.

We perform the Lorenz transformation

$$x = \gamma\ (\ \xi - \beta\ \tau)\ ,\quad t = \gamma\ (\ \tau - 2\ \beta\xi\ )\ ,\quad \beta\ \gamma\ \tau_o = x_o \qquad (3.3)$$

and obtain from (3.1)

$$\frac{\partial^2\psi(\tau,\xi)}{\partial\tau^2} - \frac{1}{2}\frac{\partial^2\psi(\tau,\xi)}{\partial\xi^2} - \psi + \psi^3 = -\frac{\chi}{\gamma}\,[\ \delta(\xi - \beta(\tau - \tau_o)) - \delta(\ \xi - \beta(\tau - \tau_o))] \qquad (3.4)$$



Rest frame of the kink   Laboratory frame

Fig.(1)

This describes the impurity, travelling with speed β towards the kink, which we assume at rest and located around $\xi = 0$, the impurity passes the kink around $\tau = \pm\tau_0$ as shown in Fig. (1).

Using the assumtion,

$$\psi(\tau,\xi) = \psi^{k}(\xi) + f_T(\xi)\,\phi_T(\xi) + \phi_L(\xi)\,f_L(\tau) + \int_{-\infty}^{\infty} \phi_q(\tau)\,f_q(\xi)\,dq \quad , \qquad (3.5)$$

we get the translation, local, and scattering mode equations. The translation mode is calculated elsewhere [2]. The equation for the local responce is given by:

$$\frac{\partial^2 \phi_L(\tau)}{\partial \tau^2} + \omega_L^2\,\phi_L(\tau) = -\frac{\chi}{\gamma}\left(\frac{3}{2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{\sinh \xi}{\cosh^2 \xi}\,[\,\delta(\xi - \beta(\tau-\tau_o)) - \delta(\xi - \beta(\tau+\tau_o))\,]\,d\xi \qquad (3.6)$$

Introducing the Fourier time transformation defined by

$$\Phi_L(\omega) = \int_{-\infty}^{\infty} \phi_L(\tau)\,e^{i\omega\tau}\,d\tau \quad , \qquad (3.7)$$

we get

$$\Phi_L(\omega) = -\frac{\chi\,(2\pi)^{-\frac{1}{2}}}{\beta\,\gamma}\left(\frac{3}{2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{\sinh \xi}{\cosh^2 \xi}\,\frac{1}{\omega_L^2 - \omega^2}\,[\,e^{-i\omega(\frac{\xi}{\beta}+\tau_o)} - e^{-i\omega(\frac{\xi}{\beta}-\tau_o)}\,]\,d\xi \qquad (3.8)$$

Integrating (3.8) we get

$$\Phi_L(\omega) = \frac{(2\pi)^{\frac{1}{2}}\chi\omega}{2\,i\,\gamma\,\beta}\left(\frac{3}{2}\right)^{\frac{1}{2}} \frac{e^{-i\omega\tau_o} - e^{i\omega\tau_o}}{(\omega_L^2 - \omega^2)\cosh\frac{\pi\omega}{2\beta}} \qquad (3.9)$$

When transforming back to the (ξ,τ) space we get, however, a very complicated expression and for bravity is not presented here. We give here only the steady state solution $\phi_L^s(\tau)$ for $\tau >> \tau_0$

$$\phi_L^s(\tau) = -\frac{\chi}{\gamma}\frac{\pi}{\beta^2}\left(\frac{3}{2}\right)^{\frac{1}{2}} \frac{\cos\omega_L(\tau-\tau_o) - \cos\omega_L(\tau+\tau_o)}{\cosh(\pi\omega_L/2\beta)} \qquad (3.10)$$

Later we are going to use this expression to calculate the energy transformed from the kink into its local oscillation.

From now on we are going to derive the radiation formula for a part of the perturbing force $\chi\,\delta(x+x_0)$ and by superposition we add the response for the other part $-\chi\,\delta(x-x_0)$. Hence, we get for the scattering mode the following equation:

$$\frac{\partial^2\phi_q(\tau)}{\partial\tau^2} + \omega_q^2\,\phi_q(\tau) = -\frac{\chi}{\gamma}\,\frac{e^{-iq\beta(\tau-\tau_o)}}{[\,2\pi(4+5q^2+q^4)\,]^{\frac{1}{2}}}$$

$$[\,3\,i\,q\tanh\beta(\tau-\tau_o) + 3\tanh^2\beta(\tau-\tau_o) - (1+q^2)\,] \qquad (3.11)$$

Now we write the amplitude of the scattering mode ϕ_q as sum of

$$\phi_q^o(\tau) = -\frac{\chi\, e^{-iq\beta(\tau-\tau_o)}}{\gamma\,(\omega_q^2 - q^2\beta^2)}\,[\,3\,i\,q\tanh\beta(\tau-\tau_o)$$

$$+\,3\tanh^2\beta(\tau-\tau_o) - (1+q^2)\,]/[4\pi(4+5q^2+q^4)]^{\frac{1}{2}}, \qquad (3.12)$$

$$\phi_q(\tau) = \phi_q^o(\tau) + \phi_q^1(\tau), \qquad (3.13)$$

$\phi_q^0(\tau)$ would already solve (3.11), if the time derivative of the terms containing $\tanh\beta(\tau-\tau_o)$ is absent. Transforming ϕ_q^o back into ξ space, we obtain

$$\phi_o^c(\xi,\tau) = \int_{\infty}^{\infty} \phi_q^o(\tau)\, f_q(\xi)\, dq$$

$$= -2\chi\gamma\{\kappa[3\tanh^2\beta(\tau-\tau_o) \mp \tanh\beta(\tau-\tau_o)]$$
$$[3\tanh^2\xi \pm 3\tanh\xi] + \kappa^2[3+3\tanh^2\beta(\tau-\tau_o)$$
$$\mp 6\tanh^2\beta(\tau-\tau_o)][3+3\tanh^2\xi \pm \tanh\xi]$$
$$+ \frac{\kappa^2\gamma}{16\gamma(1-\gamma^2)(1-4\gamma^2)}[4\gamma^2-1+3\tanh^2\beta(\tau-\tau_o)$$
$$\mp 6\gamma\tanh\beta(\tau-\tau_o)][4\gamma^2-1+\tanh^2\xi \pm 6\gamma\tanh\xi] , \qquad (3.14)$$

where

$$\kappa = e^{-|\xi-\beta(\tau-\tau_o)|} , \qquad (3.15)$$

and the upper (lower) sign in (3.14) refers to $\xi-\beta(\tau-\tau_0) > 0$ (<0) respectively. For $|\beta(\tau-\tau_0)| \to \infty$, this gives the Lorentz transformed initial and final dressing (3.2) of the impurity. Our initial condition is therefore satisfied, if the ϕ_q^1 satisfies

$$| \int_{\infty}^{\infty} \phi_q^1(\tau)\, f_q(\xi)\, dq \, | \to 0 \qquad \text{as} \quad \tau \to -\infty \quad ,$$

uniformly in . Next $\phi_q^1(\tau)$ has to satisfy the equation

$$\frac{\partial^2 \phi_q(\tau)}{\partial \tau^2} + \omega_q^2\, \phi_q^1(\tau) = \frac{\chi}{\gamma(\omega_q^2 + q^2\beta^2)} \frac{e^{-iq\beta(\tau-\tau_o)}}{[2\pi(4+5q^2+q^4)]^{\frac{1}{2}}} \{ \frac{6\, q^2\, \beta^2}{\cosh^2\beta(\tau-\tau_o)}$$
$$+9\, i\, q\beta \frac{d}{d\tau} [\operatorname{sech}^2\beta(\tau-\tau_o)] + 6\beta \frac{d}{d\tau} [\frac{\sinh\beta(\tau-\tau_o)}{\cosh^2\beta(\tau-\tau_o)}] \} \tag{3.16}$$

This equation is now solved easily to yield

$$\phi_q^1(\omega) = \frac{3\,\chi\,\pi}{\gamma\beta^2 (2\pi)^{\frac{1}{2}}} \frac{\omega(\omega^2 - q^2\beta^2)}{(\omega_q^2 - q^2\beta^2)} \frac{e^{i\omega\tau_o}}{[2\pi(4+5q^2+q^4)]^{\frac{1}{2}}}$$
$$\frac{1}{[\,\omega_q^2 - (\,\omega - i\,\varepsilon)^2\,]} \frac{1}{\sinh \frac{\pi}{2}(\frac{\omega}{\beta} + q\,)} \quad , \tag{3.17}$$

It now remains to invert $\phi_q^1(\omega)$. When inverting the Fourier time transform we give a small negative imaginary part $-i\,\varepsilon$ in the first term to satisfy the initial conditions. The poles, due to the second term ($\sinh\frac{\pi}{2}(\frac{\omega}{\beta} + q)$), contribute factors of the form $e^{-2n|\tau-\tau_o|}$ which converge absolutely. This describes, therefore, transient phenomena.

For $\tau - \tau_0 > 0$, we have from the poles $\omega = \pm\omega_q$ which give

$$\phi_q^r(\tau) = \sum_\sigma \frac{1}{2} e^{i\sigma\omega_q(\tau-\tau_o)}[2\pi(4 + 5q^2 + q^4)]^{-\frac{1}{2}}/\sinh \frac{\pi}{2}(\frac{\sigma\omega_q}{\beta} + q) \tag{3.18}$$

where $\sigma = \pm 1$.

All the transient contributions in (3.14) and in $\phi_q^1(\tau)$ decay in ξ as $(\tau - \tau_0)$ increases. The contribution to the total energy from these transient solutions vanishes asymptotically for large $(\tau - \tau_0)$. $\phi_q^r(\tau)$ does not decay in time, it consists of wave packet which moves and broadens according to the dispersion relation

$$\omega(q) = [\frac{q^2}{2} + 2]^{\frac{1}{2}} , \tag{3.19}$$

hence (3.18) gives the radiation formula.

4. RADIATION FORMULA AND ENERGY BALANCE

In this section, we shall consider some consequences of the radiation formula (3.18). For the case of impurity potential (2.4) we add the effect of the second delta function to get

$$\phi_q^r(\tau) = \frac{3\chi i}{2\gamma\beta^2} \sum_\sigma \frac{e^{i\sigma\omega_q(\tau-\tau_o)} - e^{i\sigma\omega_q(\tau+\tau_o)}}{[2\pi(4+5q^2+q^4)]^{\frac{1}{2}} \sinh(q + \sigma\omega_q/\beta)} \tag{4.1}$$

For the limiting case of very small impurity i.e $\tau_0 << 1$ we obtain

The total energy and momentum of the ϕ^4-theory is given by

$$E = R\int_{-\infty}^{\infty}[\tfrac{1}{2}\ (\frac{\partial\psi}{\partial t})^2 + \tfrac{1}{4}\ (\frac{\partial\psi}{\partial y})^2 - \tfrac{1}{2}\ \psi^2 + \tfrac{1}{4}\ \psi^4]\ dy \qquad (4.6.1)$$

$$P = \frac{R}{(2)^{\frac{1}{2}}}\int_{-\infty}^{\infty}(\frac{\partial\psi}{\partial t})(\frac{\partial\psi}{\partial y})\ dy \qquad (4.6.2)$$

From (4.6) we get for the total change in energy and momentum (both in kink rest frame), due to radiation, the following;

$$\Delta E_r = \sum_{\sigma} R\int_{-\infty}^{\infty}\omega_q^2\ |\phi(\sigma,q,\tau)|^2\ dq \qquad (4.7.1)$$

$$\Delta P_r = \sum_{\sigma} \frac{R}{(2)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\sigma\ \omega_q\, q\ |\phi(\sigma,q,\tau)\ ^2\ dq \qquad (4.7.2)$$

From (4.3), (4.7.1) and (4.7.2) we get

$$\Delta E_r = R\int_{-\infty}^{\infty}\frac{9\ \chi^2\ \tau_o^2}{4\gamma^2\ \beta^2\pi(1+q^2)}[\frac{1}{\sinh^2\frac{\pi}{2}(\omega q/\beta - q)} + \frac{1}{\sinh^2\frac{\pi}{2}(\omega_q/\beta + q)}]\ dq \qquad (4.8.1)$$

$$\Delta P_r = \frac{R}{(2)^{\frac{1}{2}}}\int_{\infty}^{\infty}\frac{36\omega q\ \ q\chi^2\tau_o^2}{\beta^2\ \ \gamma^2\pi(1+q^2)}[\frac{1}{\sinh\frac{\pi}{2}(\omega q/\beta - q)} - \frac{1}{\sinh\frac{\pi}{2}(\omega q/\beta + q)}]\ dq \qquad (4.8.2)$$

For the limiting case $\beta << 1$ (very small velocity of the kink), we get for the radiative energy and momentum,

$$\Delta E_r = R\,\frac{72(2)^{\frac{1}{4}}\ \tau_o^2}{\pi\ \gamma^2\ |\beta|^{7/2}}\ \ \chi^2\ e^{-\pi(2)^{\frac{1}{2}}(\frac{1}{\beta} - \beta\ )} \qquad (4.9.1)$$

$$\Delta P_r = \frac{R}{(2)^{\frac{1}{2}}} (\mathrm{sgn}\,\beta) \frac{144(2)^{\frac{1}{4}} \tau_o^2}{\pi \gamma^2 |\beta|^{5/2}} \quad e^{-\pi(2)^{\frac{1}{2}}(\frac{1}{\beta} - \beta)} \tag{4.9.2}$$

The energy of the radiation in the impurity rest frame comes out of the kink motion. The energy loss due to radiation can be determined from (4.9.1) and (4.9.2) by observing that ΔE_r and ΔP_r transform as 2-vector under Lorentz transformation

$$\Delta \overline{E}_r = \gamma(\Delta E_r + \beta (2)^{\frac{1}{2}} \Delta P_r)$$

$$= R \frac{72 (2)^{\frac{1}{4}} \tau_o^2}{\pi \gamma |\beta|^{7/2}} (1 - 2 \beta^2) e^{-\pi(2)^{\frac{1}{2}} (\frac{1}{\beta} - \beta)} \tag{4.10}$$

This gives the loss of kink energy due to radiation in the impurity rest frame.

The kink energy loss due to excitation of the local mode in kink rest frame is given by

$$\Delta E_L = \frac{R}{2} [(\frac{\partial \phi_L}{\partial \tau})^2 + \omega_L^2 \phi_L^2] \tag{4.11}$$

From (3.10) and (4.13) for $\tau_o << 1$ and $\beta << 1$ we get,

$$\Delta E_L = R \frac{27 \pi^2 \tau_o^2}{\gamma \beta^4} \quad \chi^2 \; e^{-\pi\omega_L/\beta} \tag{4.12}$$

Transforming to the impurity rest frame we get,

$$\Delta \overline{E}_L = R \frac{27 \pi^2 \tau_o^2}{\gamma \beta^4} \; \chi^2 \; e^{-\pi \omega_L/\beta} \tag{4.13}$$

The energy loss due to excitation of the translation mode equals zero since, $\omega_T^2 = 0$ and $\acute{\phi}_T(\infty) = 0$, and the energy for this mode obeys a similar equation as (4.11).

From (4.10) and (4.11), we get the total energy dissipated due to interaction with the impurity.

$$- \overline{E} = \Delta \overline{E}_r + \Delta \overline{E}_L$$

$$= R \frac{4 \tau_o^2}{\gamma \beta^4} \chi^2 \left[\frac{27 \pi^2}{4} e^{-\pi\omega_L/\beta} + \frac{18 (2)^{\frac{1}{4}}}{\pi} \beta^{\frac{1}{2}} e^{-\pi(2)^{\frac{1}{2}}(\frac{1}{\beta} - \beta)} \right] \quad (4.14)$$

From which we obtain the relative change in kink velocity

$$- \frac{\Delta \beta}{\beta} = \frac{6}{\beta^8} x_o^2 \chi^2 \left[\frac{27\pi^2}{4} e^{-\pi\omega_L/\beta} + \frac{18(2)^{\frac{1}{4}}}{\pi} \beta^{\frac{1}{2}} e^{-(2)^{\frac{1}{2}}(\frac{1}{\beta} - \beta)} \right] \quad (4.15)$$

A general perturbing force F(x), time independent in the laboratory frame, can be considered as a linear superposition of delta functions

$$F(x) = \int_{\infty}^{\infty} \delta(x - x_o) F(x_o) dx_o \quad (4.16)$$

Accordingly we get ϕ_q^r for a general force by using (4.1)

$$\phi_q^r(\tau) = \frac{3 i \chi}{2\gamma^2 \beta^2} \sum_{\sigma} \frac{e^{i \sigma \omega_q \tau}}{[2\pi(4+5q^2+q^4)]^{\frac{1}{2}}} \frac{G[-\sigma\omega_q/(\gamma\beta)]}{\sinh \frac{\pi}{2}(q+\omega_q/\beta)} , \quad (4.17)$$

where

$$G(x) = \int_{\infty}^{\infty} F(\tau) e^{-i x \tau} d\tau \quad (4.18)$$

It is possible to derive the radiation energy loss and change in velocity of the kink by using (4.17) and (4.18) for more realistic model of impurity potential.

5. CONCLUSION

Travelling ϕ^4-kinks subject to external forces not only react like Newtonian particles to those forces [2], but also emit persisting radiation, using a simple forst order perturbation theory. These have been demonstrated. The deceleration of the kink is mainly due to the kink energy loss:

(a) in radiation, which is the only mechanism of energy loss in case of Sine-Gordon soliton [6], and

(b) in exceiting of the local mode of the kink

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