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NEW MATHEMATICAL LAGRANGIAN FOR A DISSIPATIVE SYSTEM

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ABSTRACT

In the present study, new mathematical Lagrangian for the motion of a particle with damping linear in velocity and nonconservative force quadratic in the velocity is derived. We show that, using the derived Lagrangian, a Lagrangian and Hamiltonian formulation can be given for the considered particle motion. For such motion, the canonical momentum is not equal to the kinetic momentum and the Hamiltonian is not equal to the energy. In particular, we show that the Lagrangian and the Hamiltonian formulation of each problem given in [1] and [2] can be obtained from the present study.

1. INTRODUCTION

It is well-known [3,4] that there is a distinction between physical and mathematical Lagrangian for a dynamical system. The physical Lagrangian L is defined to be kinetic energy T minus potential energy V . Such physical Lagrangian , using Lagrange equations ,can describe the correct equations of motion only in one case, namely, the case of conservative systems. Other Lagrangians, which can give also the correct equations of motion for nonconservative systems, are called mathematical Lagrangians. For example, a particle of mass m and displacement $x(t)$ in a potential $V(x)$ is described by the physical Lagrangian L :

$$L = T - V(x) = \frac{1}{2} m \dot{x}^2 - V(x) \qquad (1)$$

which gives , using Lagrange equation :

$$\frac{d}{dt}\left\{ \frac{\partial L}{\partial \dot{x}} \right\} - \frac{\partial L}{\partial x} = 0 \qquad (2)$$

the equation of motion :

$$m \ddot{x} = - \frac{\partial V(x)}{\partial x} \qquad (3)$$

If additional damping force ,linear in the velocity $\dot{x}$, $(- m \gamma \dot{x})$,where γ is a constant, is acting on the particle, then the equation of motion of the new system will take the form :

$$m \ddot{x} = - m \gamma \dot{x} - \frac{\partial V(x)}{\partial x} \qquad (4)$$

which can not be given, using equation (2), by the physical Lagrangian $L = T - V(x)$. If the physical Lagrangian L is replaced by the mathematical Lagrangian L* [1] :

$$L^* = e^{\gamma t} [T - V(x)] = e^{\gamma t} [\frac{1}{2} m \dot{x}^2 - V(x)] \qquad (5)$$

in Lagrange equation (2),we get the correct equation of motion (4). Now, if the damping force is replaced by non-conservative force, quadratic in velocity, $(- \frac{1}{2} m \lambda \dot{x}^2)$, where λ is a constant, then the equation of motion of the new system assumes the form :

$$m \ddot{x} = - \frac{1}{2} m \lambda \dot{x}^2 - \frac{\partial V(x)}{\partial x} \tag{6}$$

and can be obtained from Lagrange equation (2) if physical Lagrangian L is replaced by the mathematical Lagrangian L* [2] , where

$$L^* = \frac{1}{2} m \dot{x}^2 e^{\lambda x} - \int_0^x e^{\lambda s} \frac{\partial V(s)}{\partial s} ds \tag{7}$$

Let $E = \frac{1}{2} m \dot{x}^2 + V(x)$, $P = dE/dt = [m \ddot{x} + \frac{\partial V(x)}{\partial x}] \dot{x}$,

$p = \frac{\partial L}{\partial \dot{x}}$; and $H = p \dot{x} - L$ be the energy , the power , the canonical momentum ; and the Hamiltonian of the system respectively. If the Hamiltonian H ,using either physical or mathematical Lagrangian L, exists then Hamilton equations

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{p} = - \frac{\partial H}{\partial x} \tag{8}$$

must satisfy the equations of motion.

Using the preceding analysis , we have :

for the system described by equation (3):

$$p = m \dot{x} \quad , \quad H = \frac{1}{2} m \dot{x}^2 + V(x) = \frac{p2}{2m} + V(x) = E \quad , \tag{9}$$

for the system described by the equation (4) :

$$p = m \dot{x} e^{\gamma t}, \quad H = e^{-\gamma t} [\frac{p^2}{2m}] + e^{\gamma t} V(x) = e^{\gamma t} E \quad , \tag{10}$$

and for the system described by equation (6) :

$$p = m \dot{x} e^{\lambda x}, \quad H = e^{-\lambda x} [\frac{p^2}{2m}] + \int_0^x e^{\lambda s} \frac{\partial V(s)}{\partial s} ds \tag{11}$$

and Hamilton equations (8) satisfy the equation of motion of each system. Hence,the nonconservative systems,described by equations (4) and (6), can also be described by using the Lagrangian and Hamiltonian formulations with appropriate mathematical Lagrangians and Hamiltonians.For those systems, the relationship H=E (which always holds for conservative systems) does not hold.

In [3,4], it is mentioned that,in order to quantize a system it is necessary to use a physical Lagrangian to obtain a physical Hamiltonian H which has a direct relationship with the energy E (H=E). In [1,2],the prementioned condition is criticized and it is proved that it is sufficient to use mathematical Lagrangian and mathematical Hamiltonian so that the system may be quantized. From the energy point of view, the system (3) is conservative with E=const ; the system (4) has power $(P = - m \gamma \dot{x}^2)$lost in all directions of motion $(\dot{x} \gtrless 0)$; and the system (6) has power $(P = - \frac{1}{2} m \lambda \dot{x}^3)$ lost in the positive direction of motion $(\dot{x} > 0)$ and gained in the opposite direction $(x < 0)$. In this paper,we consider a system with a particle of mass m and displacement x(t) in a potential V(x) and with damping linear in velocity $(- m \gamma \dot{x})$ together with nonconservative force $(- m \lambda \dot{x}^2)$. The considered system has different energy behaviour and is considered to be the generalized system from which the the systems,described by the equations (3),(4) and (6) can be isolated. We construct a mathematical Lagrangian **L*** and a mathematical Hamiltonian **H*** which satisfy the correct equation of motion. In particular, we show that the mathematical and physical Lagrangians of the prementioned systems can be attained from the new derived Lagrangian by considering the physical aspect of the system.

The new Lagrangian L* and Hamiltonian H* can be of great importance in the applications of quantum mechanics.

2. CONSTRUCTION OF THE MATHEMATICAL LAGRANGIAN L*

The equation of motion of the considered system is :

$$m \ddot{x} = - m \gamma \dot{x} - \frac{1}{2} m \lambda \dot{x}^2 - \frac{\partial V(x)}{\partial x} \qquad (12)$$

and Lagrange's equation for nonconservative system can be written in the form [5,6] :

$$\frac{d}{dt} [\frac{\partial T}{\partial \dot{x}}] - \frac{\partial T}{\partial x} = \tilde{Q} - \frac{\partial V(x)}{\partial x} \qquad (13)$$

where T and V(x) denote the kinetic energy and the potential energy of the conservative forces respectively; and $\tilde{Q}$ is the generalized force due to nonconservative forces.
Using the following relationships of the system :

$$\tilde{Q} = - m \gamma \dot{x} - \frac{1}{2} m \lambda \dot{x}^2 , \qquad (14)$$

$$T = \frac{1}{2} m \dot{x}^2 , \quad \frac{\partial T}{\partial \dot{x}} = m \dot{x} , \quad \dot{x} \frac{\partial T}{\partial \dot{x}} = 2 T \qquad (15)$$

$$\frac{\partial T}{\partial x} = \frac{\partial V(x)}{\partial \dot{x}} = 0 \qquad (16)$$

then equation (13) assumes the form :

$$\frac{d}{dt} [\frac{\partial T}{\partial \dot{x}}] + \gamma \frac{\partial T}{\partial \dot{x}} = - \lambda T - \frac{\partial V(x)}{\partial x} \qquad (16)$$

Multiplying each side of equation (16) by integrating factor $e^{\gamma t}$,we obtain

$$\frac{d}{dt} [\frac{\partial Z}{\partial \dot{x}}] + \lambda Z = - e^{\gamma t} [\frac{\partial V(x)}{\partial x}] \qquad (17)$$

where :

$$Z = T\, e^{\gamma t} = Z(\dot{x},t) \tag{18}$$

$$\dot{x}\,\frac{\partial Z}{\partial \dot{x}} = 2\,Z \tag{19}$$

Now,multiplying both sides of equation(17) and using (19), we obtain :

$$\frac{d}{dt}\left[\frac{\partial\,(Z\,e^{\lambda x})}{\partial\,\dot{x}}\right] - \lambda\,Z\,e^{\lambda x} = -\,e^{\gamma t}\left\{ e^{\lambda x}\left[\frac{\partial\,V(x)}{\partial\,x}\right]\right\} \tag{20}$$

Let us introduce the function W(x) in the form :

$$W(x) = e^{\lambda x}\left[\frac{\partial\,V(x)}{\partial\,x}\right] \tag{21}$$

and use the following property of definite integrals :

$$W(x) = \frac{\partial}{\partial\,x}\left[\int_0^x W(s)\,ds\right] \tag{22}$$

with the substitutions :

$$\frac{\partial\,(\,Z\,e^{\lambda x})}{\partial\,x} = \lambda\,Z\,e^{\lambda x}\quad,\quad \frac{\partial\,W(x)}{\partial\,\dot{x}} = 0 \tag{23}$$

to rewrite equation (20) in the form :

$$\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{x}}\left[\,Z\,e^{\lambda x} - e^{\gamma t}\int_0^x e^{\lambda s}\,\frac{\partial\,V(s)}{\partial\,s}\,ds\,\right]\right\} - \frac{\partial}{\partial\,x}\left\{Z\,e^{\lambda x} - e^{\gamma t}\int_0^x e^{\lambda s}\,\frac{\partial\,V(s)}{\partial\,s}\,ds\,\right\} = 0 \tag{24}$$

which is reduced ,using the relationships (15) and (18), to the well-known Lagrange form :

$$\frac{d}{dt}\left\{ \frac{\partial L^*}{\partial \dot{x}} \right\} - \frac{\partial L^*}{\partial x} = 0 \quad , \qquad (25)$$

with Lagrangian L^* :

$$L^* = e^{\gamma t} \left\{ \frac{1}{2} m \dot{x}^2 e^{\lambda x} - \int_0^x e^{\lambda s} \frac{\partial V(s)}{\partial s} ds \right\} \qquad (26)$$

Using the constructed mathematical Lagrangian L^* and the equation (25) we obtain the equation of motion (12) . On the other hand the physical Lagrangian L :

$$L = \frac{1}{2} m \dot{x}^2 - V(x)$$

can not satisfy the equation of motion .

3. THE HAMILTONIAN H* & HAMILTONIAN FORMULATION

Using the Lagrangian L^* ,we obtain the canonical momentum $p = \frac{\partial L^*}{\partial \dot{x}}$ from (26) :

$$P = m \dot{x} \; e^{(\gamma t + \lambda x)} \qquad (27)$$

and the Hamiltonian H^* can be attained in the usual manner:

$$H^* = \tilde{p} \dot{x} - \tilde{L}^* \qquad (28)$$

Using equations (26 - 28), we get :

$$H^* = [p^2 e^{-(\gamma t + \lambda x)} /2m] + e^{\gamma t} \int_0^x e^{\lambda s} \frac{\partial V(s)}{\partial s} ds \qquad (29)$$

Again , using the mathematical Hamiltonian H^* and Hamilton equations :

$$\dot{x} = \frac{\partial H^*}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H^*}{\partial x} \quad , \tag{30}$$

we obtain two first order differential equations satisfying the equation of motion (12). Expressing the Hamiltonian H* in terms of the energy E :

$$H^* = E^* \quad , \quad E^* = e^{\gamma t} \left\{ [E - V(x)] e^{\lambda x} + \int_0^x e^{\lambda s} \frac{\partial V(s)}{\partial s} ds \right\} \tag{31}$$

we show that Hamiltonian H* is not equivalent to the energy E of the system.

The Hamiltonian H* can ,also, be expressed as a function of $x, \dot{x}, t$ by the formula :

$$H^* = e^{\gamma t} \left\{ \frac{1}{2} m \dot{x}^2 e^{\lambda x} + \int_0^x e^{\lambda s} \frac{\partial V(s)}{\partial s} ds \right\} \tag{32}$$

The power P of the system is :

$$P = [- m \gamma \dot{x} - \frac{1}{2} m \lambda \dot{x}^2] \dot{x} \tag{33}$$

Since the damping force $(- m \gamma \dot{x})$ is always a friction force ,while the force $(- \frac{1}{2} m \lambda \dot{x}^2)$ acts as friction force if $\dot{x} > 0$ and acts as antifriction force if $\dot{x} < 0$, then the power P is always lost for $(\dot{x} > - 2\gamma / \lambda)$ and gained for $(\dot{x} < - 2\gamma / \lambda)$. This behaviour of energy is due to the interaction between the system and external world (namely, the damping medium and the medium which creates the non-conservative force). The influence of the external world for the considered system is governed by the constants $\{\gamma, \lambda\}$. If $\gamma = \lambda = 0$,then the system is isolated (conservative) and we recover the simple system,described by equation (3) for which the equatios (1) and (9), determining the Lagrangian L; the canonical momentum p; the Hamiltonian H, can be generated (by the substitution $\gamma = \lambda = 0$) from the equations (26 - 27 - 29). For such system, the Hamiltonian H is interpreted by the energy E of the system, which can be easily quantized.

If one of the media of external world, say the medium creating nonconservative force , is absent ($\lambda = 0$) and the other medium, say the damping medium, exists ($\gamma > 0$), then equation (12) generates the system (4) and the equations (26 - 27 - 29) reproduce the equations (5) and (10). For such system, the canonical momentum p is not equal to the kinetic momentum ($m\dot{x}$) and the Hamiltonian H is not equal to the energy. Using Feynman path integral method [7], the system (4) can be quantized.

Now let $\gamma = 0$ and $\lambda > 0$. In this case, equation (12) gives the system (6) and the equations (26 -27- 29) reproduce the equations (7) and (11). The particle of this system is canonically quantized in [2] by using Feynman path integral procedure to eliminate the ambiguities concerning operators ordering.

Since the Hamiltonian formalism are constructed for the considered dissipative system ,described by equation (12), then this system can be quantized [1]. The problem of quantization the considered system, using the mathematicl Lagrangian L* and the mathematical Hamiltonian H* derived in this work, will appear in seperate paper.

4. CONCLUSION

The Lagrangian and Hamiltonian formulations were, early, attained for two specific nonconservative systems, namely when either damping force - linear in velocity- **or** nonconservative force - quadratic in velocity- acts on the particle of the system. For the two systems, there are mathematical Lagrangians, which are not equivalent to the physical Lagrangians and the Hamiltonians are not equivalent to the energy of the system. However, canonical quantization of each system is possible.

The system , presented in this paper, is a more generalized system because it is assumed to include the linear - in velocity- damping force **and** the nonconservative - quadratic in vlocity-force. New mathematical Lagrangian **L*** is derived to describe the correct equation of motion of the system. The corresponding mathematical Hamiltonian **H*** is , also, determined. Complete Lagrangian and Hamiltonian formalisms are achieved. The paper also shows that the mathematical Lagrangians and Hamiltonians, early derived,for the prementioned simpler systems can be generated from derived, here,mathematical Lagrangian **L*** and Hamiltonian **H***.

Attaining the Hamiltonian formalism of generalized system, the canonical quantization is possible if Feynman path integral method is used. The quantization of the considered system is recommended for further work.

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